

THE BITANGENTIAL MATRIX NEVANLINNA-PICK INTERPOLATION PROBLEM REVISITED

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Dedicated to Heinz Langer, with respect and admiration

ABSTRACT. We revisit four approaches to the BiTangential Operator Argument Nevanlinna-Pick (BTOA-NP) interpolation theorem on the right half plane: (1) the state-space approach of Ball-Gohberg-Rodman, (2) the Fundamental Matrix Inequality approach of the Potapov school, (3) a reproducing kernel space interpretation for the solution criterion, and (4) the Grassmannian/Kreĭn-space geometry approach of Ball-Helton. These four approaches lead to three distinct solution criteria which therefore must be equivalent to each other. We give alternative concrete direct proofs of each of these latter equivalences. In the final section we show how all the results extend to the case where one seeks to characterize interpolants in the Kreĭn-Langer generalized Schur class \mathcal{S}_κ of meromorphic matrix functions on the right half plane, with the integer κ as small as possible.

1. INTRODUCTION

The simple-multiplicity case of the BiTangential Nevanlinna-Pick (BTNP) Interpolation Problem over the right half plane $\Pi_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ can be formulated as follows. Let $\mathcal{S}^{p \times m}(\Pi_+)$ denote the Schur class of $\mathbb{C}^{p \times m}$ -valued functions that are analytic and contractive-valued on Π_+ :

$$\mathcal{S}^{p \times m}(\Pi_+) := \{S : \Pi_+ \rightarrow \mathbb{C}^{p \times m} : \|S(\lambda)\| \leq 1 \text{ for all } \lambda \in \Pi_+\}.$$

The data set $\mathfrak{D}_{\text{simple}}$ for the problem consists of a collection of the form

$$\begin{aligned} \mathfrak{D}_{\text{simple}} = \{ & z_i \in \Pi_+, \ x_i \in \mathbb{C}^{1 \times p}, \ y_i \in \mathbb{C}^{1 \times m} \text{ for } i = 1, \dots, N, \\ & w_j \in \Pi_+, \ u_j \in \mathbb{C}^{m \times 1}, \ v_j \in \mathbb{C}^{p \times 1} \text{ for } j = 1, \dots, N', \\ & \rho_{ij} \in \mathbb{C} \text{ for } (i, j) \text{ such that } z_i = w_j =: \xi_{ij}\}. \end{aligned} \quad (1.1)$$

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The problem then is to find a function $S \in \mathcal{S}^{p \times m}(\Pi_+)$ that satisfies the collection of interpolation conditions

$$x_i S(z_i) = y_i \quad \text{for } i = 1, \dots, N, \quad (1.2)$$

$$S(w_j) u_j = v_j \quad \text{for } j = 1, \dots, N', \quad (1.3)$$

$$x_i S'(\xi_{ij}) y_j = \rho_{ij} \quad \text{for } (i, j) \text{ such that } z_i = w_j =: \xi_{ij}. \quad (1.4)$$

We note that the existence of a solution S to interpolation conditions (1.2), (1.3), (1.4) forces the data set (1.1) to satisfy additional compatibility equations; indeed, if S solves (1.2)–(1.4), and if (i, j) is a pair of indices where $z_i = w_j =: \xi_{ij}$, then the quantity $x_i S(\xi_{ij}) u_j$ can be computed in two ways:

$$\begin{aligned} x_i S(\xi_{ij}) u_j &= (x_i S(\xi_{ij})) u_j = y_i u_j, \\ x_i S(\xi_{ij}) u_j &= x_i (S(\xi_{ij}) u_j) = x_i v_j \end{aligned}$$

forcing the compatibility condition

$$x_i v_j = y_i u_j \quad \text{if } z_i = w_j. \quad (1.5)$$

Moreover, there is no loss of generality in assuming that each row vector x_i and each column vector u_j in (1.1) is nonzero; if $x_i = 0$ for some i , existence of a solution S then forces also that $y_i = 0$ and then the interpolation condition $x_i S(z_i) = y_i$ collapses to $0 = 0$ and can be discarded, with a similar analysis in case some $u_j = 0$.

The following result gives the precise solution criterion. The result actually holds even without the normalization conditions on the data set discussed in the previous paragraph.

Theorem 1.1. (See [36, Section 4] for the case where $z_i \neq w_j$ for all i, j). *Given a data set $\mathfrak{D}_{\text{simple}}$ as in (1.1), there exists a solution S of the associated problem BTNP if and only if the associated Pick matrix*

$$P_{\mathfrak{D}_{\text{simple}}} := \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \quad (1.6)$$

with entries given by

$$\begin{aligned} [P_{11}]_{ij} &= \frac{x_i x_j^* - y_i y_j^*}{z_i + \bar{z}_j} \quad \text{for } 1 \leq i, j \leq N, \\ [P_{12}]_{ij} &= \begin{cases} \frac{x_i v_j - y_i u_j}{w_j - z_i} & \text{if } z_i \neq w_j, \\ \rho_{ij} & \text{if } z_i = w_j, \end{cases} \quad \text{for } 1 \leq i \leq N, 1 \leq j \leq N', \\ [P_{22}]_{ij} &= \frac{u_i^* u_j - v_i^* v_j}{\bar{w}_i + w_j} \quad \text{for } 1 \leq i, j \leq N', \end{aligned}$$

is positive semidefinite.

Given a data set $\mathfrak{D}_{\text{simple}}$ as above, it is convenient to repackage it in a more aggregate form as follows (see [9]). With data as in (1.1), form the

septet of matrices $(Z, X, Y, W, U, V, \Gamma)$ where:

$$\begin{aligned} Z &= \begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_N \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \\ W &= \begin{bmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_{N'} \end{bmatrix}, \quad U = [u_1 \quad \cdots \quad u_{N'}], \quad V = [v_1 \quad \cdots \quad v_{N'}], \\ \Gamma &= [\gamma_{ij}]_{i=1, \dots, N}^{j=1, \dots, N'} \quad \text{where} \quad \gamma_{ij} = \begin{cases} \frac{x_i v_j - y_i u_j}{w_j - z_i} & \text{if } z_i \neq w_j, \\ \rho_{ij} & \text{if } z_i = w_j. \end{cases} \end{aligned} \quad (1.7)$$

Note that the compatibility condition (1.5) translates to the fact that Γ satisfies the Sylvester equation

$$\Gamma W - Z \Gamma = [X \quad -Y] \begin{bmatrix} V \\ U \end{bmatrix}.$$

The normalization requirements ($x_i \neq 0$ for all i and $u_j \neq 0$ for all j together with z_1, \dots, z_N all distinct and $w_1, \dots, w_{N'}$ all distinct) translate to the conditions

$$(Z, X) \text{ is controllable,} \quad (U, W) \text{ is observable.}$$

Then it is not hard to see that the interpolation conditions (1.2), (1.3), (1.4) can be written in the more aggregate form

$$\sum_{z_0 \in \sigma(Z)} \text{Res}_{\lambda=z_0} (\lambda I - Z)^{-1} X S(\lambda) = Y, \quad (1.8)$$

$$\sum_{z_0 \in \sigma(W)} \text{Res}_{\lambda=z_0} S(\lambda) U (\lambda I - W)^{-1} = V, \quad (1.9)$$

$$\sum_{z_0 \in \sigma(Z) \cup \sigma(W)} \text{Res}_{\lambda=z_0} (\lambda I - Z)^{-1} X S(\lambda) U (\lambda I - W)^{-1} = \Gamma. \quad (1.10)$$

Suppose that (Z, X) is any controllable input pair and that (U, W) is an observable output pair. Assume in addition that $\sigma(Z) \cup \sigma(W) \subset \Pi_+$ and that S is an analytic matrix function (of appropriate size) on Π_+ . We define the **Left-Tangential Operator Argument (LTOA) point evaluation** $(XS)^{\wedge L}(Z)$ of S at Z in left direction X by

$$(XS)^{\wedge L}(Z) = \sum_{z_0 \in \sigma(Z)} \text{Res}_{\lambda=z_0} (\lambda I - Z)^{-1} X S(\lambda).$$

Similarly we define the **Right-Tangential Operator Argument (RTOA) point evaluation** $(SU)^{\wedge R}(W)$ of S at W in right direction U by

$$(SU)^{\wedge R}(W) = \sum_{z_0 \in \sigma(W)} \text{Res}_{\lambda=z_0} S(\lambda) U (\lambda I - W)^{-1}.$$

Finally the **BiTangential Operator Argument (BTOA) point evaluation** $(XSU)^{\wedge L,R}(Z, W)$ of S at left argument Z and right argument W in left direction X and right direction U is given by

$$(XSU)^{\wedge L,R}(Z, W) = \sum_{z_0 \in \sigma(Z) \cup \sigma(W)} \text{Res}_{\lambda=z_0} (\lambda I - Z)^{-1} X S(\lambda) U (\lambda I - W)^{-1}.$$

With this condensed notation, we write the interpolation conditions (1.8), (1.9), (1.10) simply as

$$(XS)^{\wedge L}(Z) = Y, \quad (1.11)$$

$$(SU)^{\wedge R}(W) = V, \quad (1.12)$$

$$(XSU)^{\wedge L,R}(Z, W) = \Gamma. \quad (1.13)$$

Let us say that the data set

$$\mathfrak{D} = (Z, X, Y; U, V, W; \Gamma) \quad (1.14)$$

is a Π_+ -admissible *BiTangential Operator Argument (BTOA) interpolation data set* if the following conditions hold:

- (1) Both Z and W have spectrum inside Π_+ : $\sigma(Z) \cup \sigma(W) \subset \Pi_+$.
- (2) (Z, X) is controllable and (U, W) is observable.
- (3) Γ satisfies the Sylvester equation

$$\Gamma W - Z\Gamma = XV - YU. \quad (1.15)$$

Then it makes sense to consider the collection of interpolation conditions (1.11), (1.12), (1.13) for *any* Π_+ -admissible BTOA interpolation data set $(Z, X, Y; U, V, W; \Gamma)$. It can be shown that these interpolation conditions can be expressed equivalently as a set of higher-order versions of the interpolation conditions (1.2), (1.3), (1.4) (see [9, Theorem 16.8.1]), as well as a representation of S in the so-called Model-Matching form (see [9, Theorem 16.9.3], [25])

$$S(\lambda) = T_1(\lambda) + T_2(\lambda)Q(\lambda)T_3(\lambda),$$

where T_1, T_2, T_3 are rational matrix functions analytic on Π_+ with T_2 and T_3 square and analytic and invertible along the imaginary line, and where Q is a free-parameter matrix function analytic on all of Π_+ .

It is interesting to note that the Sylvester equation (1.15) is still necessary for the existence of a $p \times m$ -matrix function S analytic on Π_+ satisfying the BTOA interpolation conditions (1.11), (1.12), (1.13). Indeed, note that

$$\begin{aligned} & ((\lambda I - Z)^{-1} X S(\lambda) U (\lambda I - W)^{-1}) W - Z ((\lambda I - Z)^{-1} X S(\lambda) U (\lambda I - W)^{-1}) \\ &= (\lambda I - Z)^{-1} X S(\lambda) U (\lambda I - W)^{-1} (W - \lambda I + \lambda I) \\ &\quad + (\lambda I - Z - \lambda I) (\lambda I - Z)^{-1} X S(\lambda) U (\lambda I - W)^{-1} \\ &= -(\lambda I - Z)^{-1} X S(\lambda) U + \lambda \cdot (\lambda I - Z)^{-1} X S(\lambda) U (\lambda I - W)^{-1} \\ &\quad + X S(\lambda) U (\lambda I - W)^{-1} - \lambda \cdot (\lambda I - Z)^{-1} X S(\lambda) U (\lambda I - W)^{-1} \\ &= -(\lambda I - Z)^{-1} X S(\lambda) U + X S(\lambda) U (\lambda I - W)^{-1}. \end{aligned}$$

If we now take the sum of the residues of the first and last expression in this chain of equalities over points $z_0 \in \Pi_+$ and use the interpolation conditions (1.8)–(1.10), we arrive at

$$\Gamma W - Z\Gamma = -YU + XV$$

and the Sylvester equation (1.15) follows.

We now pose the **BiTangential Operator Argument Nevanlinna-Pick (BTOA-NP) Interpolation Problem**: Given a Π_+ -admissible BTOA interpolation data set (1.14), find S in the matrix Schur class over the right half plane $\mathcal{S}^{p \times m}(\Pi_+)$ which satisfies the BTOA interpolation conditions (1.11), (1.12), (1.13).

Before formulating the solution, we need some additional notation. Given a Π_+ -admissible BTOA interpolation data set (1.14), introduce two additional matrices Γ_L and Γ_R as the unique solutions of the respective Lyapunov equations

$$\Gamma_L Z^* + Z\Gamma_L = XX^* - YY^*, \quad (1.16)$$

$$\Gamma_R W + W^*\Gamma_R = U^*U - V^*V. \quad (1.17)$$

We define the BTOA-Pick matrix $\mathbf{\Gamma}_{\mathfrak{D}}$ associated with the data set (1.14) by

$$\mathbf{\Gamma}_{\mathfrak{D}} = \begin{bmatrix} \Gamma_L & \Gamma \\ \Gamma^* & \Gamma_R \end{bmatrix}. \quad (1.18)$$

The following is the canonical generalization of Theorem 1.1 to this more general situation.

Theorem 1.2. *Suppose that*

$$\mathfrak{D} = (X, Y, Z; U, V, W; \Gamma)$$

is a Π_+ -admissible BTOA interpolation data set. Then there exists a solution $S \in \mathcal{S}^{p \times m}(\Pi_+)$ of the BTOA-NP interpolation problem associated with data set \mathfrak{D} if and only if the associated BTOA-Pick matrix $\mathbf{\Gamma}_{\mathfrak{D}}$ defined by (1.18) is positive semidefinite.

In case $\mathbf{\Gamma}_{\mathfrak{D}}$ is strictly positive definite ($\mathbf{\Gamma}_{\mathfrak{D}} \succ 0$), the set of all solutions is parametrized as follows. Define a $(p+m) \times (p+m)$ -matrix function

$$\Theta(\lambda) = \begin{bmatrix} \Theta_{11}(\lambda) & \Theta_{12}(\lambda) \\ \Theta_{21}(\lambda) & \Theta_{22}(\lambda) \end{bmatrix}$$

via

$$\Theta(\lambda) = \begin{bmatrix} I_p & 0 \\ 0 & I_m \end{bmatrix} + \begin{bmatrix} -X^* & V \\ -Y^* & U \end{bmatrix} \begin{bmatrix} \lambda I + Z^* & 0 \\ 0 & \lambda I - W \end{bmatrix}^{-1} \mathbf{\Gamma}_{\mathfrak{D}}^{-1} \begin{bmatrix} X & -Y \\ -V^* & U^* \end{bmatrix}. \quad (1.19)$$

Then S is a solution of the BTOA-NP interpolation problem if and only if S has a representation as

$$S(\lambda) = (\Theta_{11}(\lambda)G(\lambda) + \Theta_{12}(\lambda))(\Theta_{21}(\lambda)G(\lambda) + \Theta_{22}(\lambda))^{-1} \quad (1.20)$$

where G is a free-parameter function in the Schur class $\mathcal{S}^{p \times m}(\Pi_+)$.

Note that the first part of Theorem 1.2 for the special case where the data set \mathfrak{D} has the form (1.7) coming from the data set (1.1) for a BT-NP problem amounts to the content of Theorem 1.1.

The BTOA-NP interpolation problem and closely related problems have been studied and analyzed using a variety of methodologies by number of authors, especially in the 1980s and 1990s, largely inspired by connections with the then emerging H^∞ -control theory (see [25]). We mention in particular the Schur-algorithm approach in [36, 19, 2], the method of Fundamental Matrix Inequalities by the Potapov school (see e.g., [32]) and the related formalism of the Abstract Interpolation Problem of Katsnelson-Kheifets-Yuditskii (see [30, 31]), the Commutant Lifting approach of Foias-Frazho-Gohberg-Kaashoek (see [23, 24], and the Reproducing Kernel approach of Dym and collaborators (see [21, 22]). Our focus here is to revisit two other approaches: (1) the Grassmannian/Kreĭn-space-geometry approach of Ball-Helton [10], and (2) the state-space implementation of this approach due to Ball-Gohberg-Rodman ([9]). The first (Grassmannian) approach relies on Kreĭn-space geometry to arrive at the existence of a solution; the analysis is constructive only after one introduces bases to coordinatize various subspaces and operators. The second (state-space) approach has the same starting point as the first (encoding the problem in terms of the graph of the sought-after solution rather than in terms of the solution itself), but finds state-space coordinates in which to coordinatize the J -inner function parametrizing the set of solutions and then verifies the linear-fractional parametrization by making use of intrinsic properties of J -inner functions together with an explicit winding-number argument, thereby bypassing any appeal to general results from Kreĭn-space geometry. This second approach proved to be more accessible to users (e.g., engineers) who were not comfortable with the general theory of Kreĭn spaces.

It turns out that the solution criterion $\mathbf{\Gamma}_{\mathfrak{D}} \succeq 0$ arises more naturally in the second (state-space) approach. Furthermore, when $\mathbf{\Gamma}_{\mathfrak{D}} \succ 0$ ($\mathbf{\Gamma}_{\mathfrak{D}}$ is strictly positive definite), one gets a linear-fractional parametrization for the set of all Schur-class solutions of the interpolation conditions. The matrix function Θ generating the linear-fractional map also generates a matrix kernel function $K_{\Theta,J}$ which is a positive kernel exactly when $\mathbf{\Gamma}_{\mathfrak{D}} \succ 0$. We can then view the fact that the associated reproducing kernel space $\mathcal{H}(K_{\Theta,J})$ is a Hilbert space as also a solution criterion for the BTOA-NP interpolation problem in the nondegenerate case.

In the first (Grassmannian/Kreĭn-space-geometry) approach, on the other hand, the immediate solution criterion is in terms of the positivity of a certain finite-dimensional subspace $(\mathcal{M}_{\mathfrak{D}}^{[\perp\mathcal{K}]})_0$ of a Kreĭn space constructed from the interpolation data \mathfrak{D} . In the Left Tangential case, one can identify $\mathbf{\Gamma}_{\mathfrak{D}}$ as the Kreĭn-space gramian matrix with respect to a natural basis for $(\mathcal{M}_{\mathfrak{D}}^{[\perp\mathcal{K}]})_0$, thereby confirming directly the equivalence of the two seemingly distinct solution criteria. For the general BiTangential case, the connection

between $\Gamma_{\mathfrak{D}}$ and $(\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]})_0$ is not so direct, but nevertheless, using ideas from [12], we present here a direct proof as to why $\Gamma_{\mathfrak{D}} \succeq 0$ is equivalent to Kreĭn-space positivity of $(\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]})_0$ which is interesting in its own right. Along the way, we also show how the Fundamental Matrix Inequality approach to interpolation of the Potapov school [32] can be incorporated into this BTOA-interpolation formalism to give an alternative derivation of the linear-fractional parametrization which also bypasses the winding-number argument, at least for the classical Schur-class setting. We also sketch how all the results extend to the more general problem where one seeks solutions of the BTOA interpolation conditions (1.11)–(1.13) in the Kreĭn-Langer generalized Schur class $\mathcal{S}_{\kappa}^{p \times m}(\Pi_+)$ with the integer κ as small as possible.

The plan of the paper is as follows. In Section 2 we sketch the ideas of the second (state-space) approach, with the Fundamental Matrix Inequality approach and the reproducing-kernel interpretation dealt with in succeeding subsections. In Section 3 we sketch the somewhat more involved ideas behind the first (Grassmannian/Kreĭn-space-geometry) approach. In Section 4 we identify the connections between the two approaches and in particular show directly that the two solution criteria are indeed equivalent. In the final Section 5 we indicate how the setup extends to interpolation problems for the generalized Schur class $\mathcal{S}_{\kappa}^{p \times m}(\Pi_+)$.

2. THE STATE-SPACE APPROACH TO THE BTOA-NP INTERPOLATION PROBLEM

In this section we sketch the analytic proof of Theorem 1.2 from [9]. For \mathcal{U} and \mathcal{V} Hilbert spaces, we let $\mathcal{L}(\mathcal{U}, \mathcal{V})$ denote the space of bounded linear operators mapping \mathcal{U} into \mathcal{V} , abbreviated to $\mathcal{L}(\mathcal{U})$ in case $\mathcal{U} = \mathcal{V}$. We then define the operator-valued version of the Schur class $\mathcal{S}_{\Omega}(\mathcal{U}, \mathcal{V})$ to consist of holomorphic functions S on Ω with values equal to contraction operators between \mathcal{U} and \mathcal{V} .

We first recall some standard facts concerning positive kernels and reproducing kernel Hilbert spaces (see e.g., [7]). Given a point-set Ω and coefficient Hilbert space \mathcal{V} along with a function $K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{V})$, we say that K is a *positive kernel on Ω* if

$$\sum_{i,j=1}^N \langle K(\omega_i, \omega_j) y_j, y_i \rangle_{\mathcal{V}} \geq 0 \quad (2.1)$$

for any collection of N points $\omega_1, \dots, \omega_N \in \Omega$ and vectors $y_1, \dots, y_N \in \mathcal{V}$ with arbitrary $N \geq 1$. It is well known that the following are equivalent:

- (1) K is a positive kernel on Ω .
- (2) K is the *reproducing kernel* for a *reproducing kernel Hilbert space* $\mathcal{H}(K)$ consisting of functions $f: \Omega \rightarrow \mathcal{V}$ such that, for each $\omega \in \Omega$ and $y \in \mathcal{V}$ the function $k_{\omega, y}: \Omega \rightarrow \mathcal{V}$ defined by

$$k_{\omega, y}(\omega') = K(\omega', \omega)y \quad (2.2)$$

is in $\mathcal{H}(\Omega)$ and has the reproducing property: for each $f \in \mathcal{H}(K)$,

$$\langle f, k_{\omega, y} \rangle_{\mathcal{H}(K)} = \langle f(\omega), y \rangle_{\mathcal{Y}}. \quad (2.3)$$

- (3) K has a *Kolmogorov decomposition*: there is a Hilbert space \mathcal{X} and a function $H: \Omega \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})$ so that

$$K(\omega', \omega) = H(\omega')H(\omega)^*. \quad (2.4)$$

Proof of Theorem 1.2. We first illustrate the proof of necessity for the easier simple-multiplicity case as formulated in Theorem 1.1; the idea is essentially the same as the necessity proof in Limebeer-Anderson [36].

It is well known that a Schur-class function $F \in \mathcal{S}_{\mathbb{D}}(\mathcal{U}, \mathcal{Y})$ on the unit disk can be characterized not only by the positivity of the de Branges-Rovnyak kernel

$$\mathbf{K}_F(\lambda, w) = \frac{I - F(\lambda)F(w)^*}{1 - \lambda \bar{w}}$$

on the unit disk \mathbb{D} , but also by positivity of the block 2×2 -matrix kernel defined on $(\mathbb{D} \times \mathbb{D}) \times (\mathbb{D} \times \mathbb{D})$ by

$$\tilde{\mathbf{K}}_F(\lambda, \lambda_*; w, w_*) := \begin{bmatrix} \frac{I - F(\lambda)F(w)^*}{1 - \lambda \bar{w}} & \frac{F(\lambda) - F(\bar{w}_*)}{\lambda - \bar{w}_*} \\ \frac{F(\bar{\lambda}_*)^* - F(w)^*}{\lambda_* - \bar{w}} & \frac{I - F(\bar{\lambda}_*)^*F(\bar{w}_*)}{1 - \lambda_* \bar{w}_*} \end{bmatrix}.$$

Making use of the linear-fractional change of variable from \mathbb{D} to Π_+

$$\lambda \in \mathbb{D} \mapsto z = \frac{1 + \lambda}{1 - \lambda} \in \Pi_+$$

with inverse given by

$$z \in \Pi_+ \mapsto \lambda = \frac{z - 1}{z + 1} \in \mathbb{D},$$

it is easily seen that the function S defined on Π_+ is in the Schur class $\mathcal{S}_{\Pi_+}(\mathcal{U}, \mathcal{Y})$ over Π_+ if and only if, not only the Π_+ -de Branges-Rovnyak kernel

$$K_S(z, \zeta) = \frac{I - S(z)S(\zeta)^*}{z + \bar{\zeta}} \quad (2.5)$$

is a positive kernel on Π_+ , but also the (2×2) -block de Branges-Rovnyak kernel

$$\mathbf{K}_S(z, z_*; \zeta, \zeta_*) := \begin{bmatrix} \frac{I - S(z)S(\zeta)^*}{z + \bar{\zeta}} & \frac{S(z) - S(\bar{\zeta}_*)}{z - \bar{\zeta}_*} \\ \frac{S(\bar{z}_*)^* - S(\zeta)^*}{z_* - \bar{\zeta}} & \frac{I - S(\bar{z}_*)^*S(\bar{\zeta}_*)}{z_* + \bar{\zeta}_*} \end{bmatrix} \quad (2.6)$$

is a positive kernel on $(\Pi_+ \times \Pi_+) \times (\Pi_+ \times \Pi_+)$. Specifying the latter kernel at the points $(z, z_*), (\zeta, \zeta_*) \in \Pi_+ \times \Pi_+$ where $z, \zeta = z_1, \dots, z_N$ and $z_*, \zeta_* =$

$\bar{w}_1, \dots, \bar{w}_{N'}$, leads to the conclusion that the block matrix

$$\begin{bmatrix} \left[\frac{I - S(z_i)S(z_j)^*}{z_i + \bar{z}_j} \right] & \left[\frac{S(z_i) - S(w_{j'})}{z_i - w_{j'}} \right] \\ \left[\frac{S(w_{i'})^* - S(z_j)^*}{\bar{w}_{j'} - \bar{z}_i} \right] & \left[\frac{I - S(w_{i'})^*S(w_{j'})}{\bar{w}_{i'} + w_{j'}} \right] \end{bmatrix}, \quad (2.7)$$

where $1 \leq i, j \leq N$ and $1 \leq i', j' \leq N'$, is positive semidefinite. Note that the entry $\frac{S(z_i) - S(w_{j'})}{z_i - w_{j'}}$ in the upper right corner is to be interpreted as $S'(\xi_{ij'})$ in case $z_i = w_{j'} =: \xi_{i,j'}$ for some pair of indices i, j' .

Suppose now that $S \in \mathcal{S}_{\Pi_+}(\mathcal{U}, \mathcal{Y})$ is a Schur-class solution of the interpolation conditions (1.2), (1.3), (1.4). When we multiply the matrix (2.7) on the left by the block diagonal matrix

$$\begin{bmatrix} \text{diag}_{1 \leq i \leq N} [x_i] & 0 \\ 0 & \text{diag}_{1 \leq i' \leq N'} [u_{i'}^*] \end{bmatrix}$$

and on the right by its adjoint, we arrive at the matrix $P_{\mathfrak{D}_{\text{simple}}}$. This verifies the necessity of the condition $P_{\mathfrak{D}_{\text{simple}}} \succeq 0$ for a solution of the BT-NP interpolation problem to exist.

We now consider the proof of necessity for the general case. We note that the proof of necessity in [9] handles explicitly only the case where the Pick matrix is invertible and relies on use of the matrix-function Θ generating the linear-fractional parametrization (see (2.17) below). We give a proof here which proceeds directly from the BTOA-interpolation formulation; it amounts to a specialization of the proof of necessity for the more complicated multivariable interpolation problems in the Schur-Agler class done in [6].

The starting point is the observation that the positivity of the kernel \mathbf{K}_S implies that it has a Kolmogorov decomposition (2.4); furthermore the extra structure of the arguments of the kernel \mathbf{K}_S implies that the Kolmogorov decomposition can be taken to have the form

$$\mathbf{K}_S(z, z_*; \zeta, \zeta_*) = \begin{bmatrix} H(z) \\ G(z_*)^* \end{bmatrix} \begin{bmatrix} H(\zeta)^* & G(\zeta_*) \end{bmatrix} \quad (2.8)$$

for holomorphic operator functions

$$H: \Pi_+ \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y}), \quad G: \Pi_+ \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{X}).$$

In the present matricial setting of $\mathbb{C}^{p \times m}$ -valued functions, the spaces \mathcal{U} and \mathcal{Y} are finite dimensional and can be identified with \mathbb{C}^p and \mathbb{C}^m , respectively. In particular we read off the identity

$$H(z)G(\zeta) = \frac{S(z) - S(\zeta)}{z - \zeta} \quad (2.9)$$

with appropriate interpretation in case $z = \zeta$. Observe that for a fixed $\zeta \in \Pi_+ \setminus \sigma(Z)$, we have from (2.9)

$$\begin{aligned}
(XH)^{\wedge L}(Z) \cdot G(\zeta) &= \sum_{z_0 \in \sigma(Z)} \text{Res}_{\lambda=z_0} (\lambda I - Z)^{-1} XH(\lambda)G(\zeta) \\
&= \sum_{z_0 \in \sigma(Z)} \text{Res}_{\lambda=z_0} (\lambda I - Z)^{-1} X \frac{S(\lambda) - S(\zeta)}{\lambda - \zeta} \\
&= \sum_{z_0 \in \sigma(Z)} \text{Res}_{\lambda=z_0} (\lambda I - Z)^{-1} \frac{Y - XS(\zeta)}{\lambda - \zeta} \\
&= (\zeta I - Z)^{-1} (XS(\zeta) - Y)
\end{aligned} \tag{2.10}$$

where we used the interpolation condition (1.8) for the third equality. Since the function $g(\zeta) = (\zeta I - Z)^{-1} YU(\zeta I - W)^{-1}$ satisfies an estimate of the form $\|g(\zeta)\| \leq \frac{M}{|\zeta|^2}$ as $|\zeta| \rightarrow \infty$, it follows that

$$\sum_{z_0 \in \sigma(Z) \cup \sigma(W)} \text{Res}_{\zeta=z_0} (\zeta I - Z)^{-1} YU(\zeta I - W)^{-1} = 0.$$

On the other hand, due to condition (1.8), the function on the right hand side of (2.10) is analytic (in ζ) on Π_+ , so that

$$\begin{aligned}
&\sum_{z_0 \in \sigma(W)} \text{Res}_{\zeta=z_0} (\zeta I - Z)^{-1} (XS(\zeta) - Y)U(\zeta I - W)^{-1} \\
&= \sum_{z_0 \in \sigma(Z) \cup \sigma(W)} \text{Res}_{\zeta=z_0} (\zeta I - Z)^{-1} (XS(\zeta) - Y)U(\zeta I - W)^{-1}.
\end{aligned}$$

We now apply the **RTOA** point evaluation to both sides in (2.10) and make use of the two last equalities and the interpolation condition (1.10):

$$\begin{aligned}
&(XH)^{\wedge L}(Z)(GU)^{\wedge R}(W) \\
&= \sum_{z_0 \in \sigma(W)} \text{Res}_{\zeta=z_0} (\zeta I - Z)^{-1} (XS(\zeta) - Y)U(\zeta I - W)^{-1} \\
&= \sum_{z_0 \in \sigma(Z) \cup \sigma(W)} \text{Res}_{\zeta=z_0} (\zeta I - Z)^{-1} (XS(\zeta) - Y)U(\zeta I - W)^{-1} \\
&= \sum_{z_0 \in \sigma(Z) \cup \sigma(W)} \text{Res}_{\zeta=z_0} (\zeta I - Z)^{-1} XS(\zeta)U(\zeta I - W)^{-1} = \Gamma.
\end{aligned} \tag{2.11}$$

Let us now introduce the block 2×2 -matrix $\mathbf{\Gamma}'_{\mathfrak{D}}$ by

$$\mathbf{\Gamma}'_{\mathfrak{D}} = \begin{bmatrix} (XH)^{\wedge L}(Z) \\ (GU)^{\wedge R}(W)^* \end{bmatrix} \begin{bmatrix} ((XH)^{\wedge L}(Z))^* & (GU)^{\wedge R}(W) \end{bmatrix}. \tag{2.12}$$

We claim that $\Gamma'_{\mathfrak{D}} = \Gamma_{\mathfrak{D}}$. Note that equality of the off-diagonal blocks follows from (2.11). It remains to show the two equalities

$$\Gamma'_L := (XH)^{\wedge L}(Z)((XH)^{\wedge L}(Z))^* = \Gamma_L, \quad (2.13)$$

$$\Gamma'_R := ((G^*U)^{\wedge R}(W))^*(G^*U)^{\wedge R}(W) = \Gamma_R. \quad (2.14)$$

To verify (2.13), we note that Γ_L is defined as the unique solution of the Lyapunov equation (1.16). Thus it suffices to verify that Γ'_L also satisfies (1.16). Toward this end, the two expressions (2.6) and (2.8) for \mathbf{K}_S give us equality of the $(1,1)$ -block entries:

$$H(z)H(\zeta)^* = \frac{I - S(z)S(\zeta)^*}{z + \bar{\zeta}}$$

which we prefer to rewrite in the form

$$z \cdot H(z)H(\zeta)^* + H(z)H(\zeta)^* \cdot \bar{\zeta} = I - S(z)S(\zeta)^*. \quad (2.15)$$

To avoid confusion, let us introduce the notation χ for the identity function $\chi(z) = z$ on Π_+ . Then it is easily verified that

$$(X\chi \cdot H)^{\wedge L}(Z) = Z(XH)^{\wedge L}(Z). \quad (2.16)$$

Multiplication on the left by X and on the right by X^* and then plugging in the left operator argument Z for λ in (2.15) then gives

$$\begin{aligned} Z(XH)^{\wedge L}(Z)H(\zeta)^*X^* + (XH)^{\wedge L}(Z)(\zeta \cdot H(\zeta)^*X^* \\ = XX^* - (XS)^{\wedge L}(Z)S(\zeta)^* = XX^* - YS(\zeta)^*X^*. \end{aligned}$$

Replacing the variable ζ by the operator argument Z and applying the adjoint of the identity (2.16) then brings us to

$$Z(XH)^{\wedge L}(Z)((XH)^{\wedge L}(Z))^*Z^* = XX^* - Y(XS^{\wedge L}(Z))^* = XX^* - YY^*,$$

i.e., Γ'_L satisfies (1.16) as wanted. The proof that Γ'_R (see (2.14)) satisfies (1.17) proceeds in a similar way.

For the sufficiency direction, for simplicity we shall assume that $\Gamma_{\mathfrak{D}}$ is strictly positive definite rather than just positive semidefinite. We then must show that solutions S of the BTOA-NP problem exist and in fact the set of all solutions is given by the linear-fractional parametrization (1.20). The case where the Pick matrix is positive-semidefinite then follows by perturbing the semidefinite Pick matrix to a definite Pick matrix and using an approximation and normal families argument. The ideas follow [9].

Let us therefore assume that $\Gamma_{\mathfrak{D}}$ is positive definite. Then we can form the rational matrix function Θ given by (1.19). Let us write Θ in the more condensed form

$$\Theta(\lambda) = I_{p+m} - \mathbf{C}(\lambda I - \mathbf{A})^{-1}\Gamma_{\mathfrak{D}}^{-1}\mathbf{C}^*J \quad (2.17)$$

where we set

$$\mathbf{A} = \begin{bmatrix} -Z^* & 0 \\ 0 & W \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -X^* & V \\ -Y^* & U \end{bmatrix}, \quad J = \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix}. \quad (2.18)$$

Recall that Γ_L , Γ_R , Γ satisfy the Lyapunov/Sylvester equations (1.16), (1.17), (1.15). Consequently one can check that $\Gamma_{\mathfrak{D}}$ satisfies the (2×2) -block Lyapunov/Sylvester equation

$$\begin{aligned} & \begin{bmatrix} \Gamma_L & \Gamma \\ \Gamma^* & \Gamma_R \end{bmatrix} \begin{bmatrix} -Z^* & 0 \\ 0 & W \end{bmatrix} + \begin{bmatrix} -Z & 0 \\ 0 & W^* \end{bmatrix} \begin{bmatrix} \Gamma_L & \Gamma \\ \Gamma^* & \Gamma_R \end{bmatrix} \\ &= \begin{bmatrix} YY^* - XX^* & XV - YU \\ V^*X^* - U^*Y^* & V^*U - V^*V \end{bmatrix}, \end{aligned}$$

or, in more succinct form,

$$\Gamma_{\mathfrak{D}}\mathbf{A} + \mathbf{A}^*\Gamma_{\mathfrak{D}} = -\mathbf{C}^*J\mathbf{C}. \quad (2.19)$$

Using this we compute

$$\begin{aligned} J - \Theta(\lambda)J\Theta(\zeta)^* &= \\ J - (I - \mathbf{C}(\lambda I - \mathbf{A})^{-1}\Gamma_{\mathfrak{D}}^{-1}\mathbf{C}^*J) J (I - J\mathbf{C}\Gamma_{\mathfrak{D}}^{-1}(\bar{\zeta}I - \mathbf{A}^*)^{-1}\mathbf{C}^*) &= \\ = \mathbf{C}(\lambda I - \mathbf{A})^{-1}\Gamma_{\mathfrak{D}}^{-1}\mathbf{C}^* + \mathbf{C}\Gamma_{\mathfrak{D}}^{-1}(\bar{\zeta}I - \mathbf{A}^*)^{-1}\mathbf{C}^* &= \\ - \mathbf{C}(\lambda I - \mathbf{A})^{-1}\Gamma_{\mathfrak{D}}^{-1}\mathbf{C}^*J\mathbf{C}\Gamma_{\mathfrak{D}}^{-1}(\bar{\zeta}I - \mathbf{A}^*)^{-1}\mathbf{C}^* &= \\ = \mathbf{C}(\lambda I - \mathbf{A})^{-1}\Gamma_{\mathfrak{D}}^{-1}\Xi(\lambda, \zeta)(\bar{\zeta}I - \mathbf{A}^*)^{-1}\Gamma_{\mathfrak{D}}^{-1}\mathbf{C}^* \end{aligned}$$

where

$$\Xi(\lambda, \zeta) = (\bar{\zeta}I - \mathbf{A}^*)\Gamma_{\mathfrak{D}} + \Gamma_{\mathfrak{D}}(\lambda I - \mathbf{A}) - \mathbf{C}^*J\mathbf{C} = (\lambda + \bar{\zeta})\Gamma_{\mathfrak{D}},$$

where we used (2.19) in the last step. We conclude that

$$K_{\Theta, J}(\lambda, \zeta) := \frac{J - \Theta(\lambda)J\Theta(\zeta)^*}{\lambda + \bar{\zeta}} = \mathbf{C}(\lambda I - \mathbf{A})^{-1}\Gamma_{\mathfrak{D}}^{-1}(\bar{\zeta}I - \mathbf{A}^*)^{-1}\mathbf{C}^*. \quad (2.20)$$

By assumption, $\sigma(Z) \cup \sigma(W) \subset \Pi_+$, so the matrix $\mathbf{A} = \begin{bmatrix} -Z^* & 0 \\ 0 & W \end{bmatrix}$ has no eigenvalues on the imaginary line, and hence Θ is analytic and invertible on $i\mathbb{R}$. As a consequence of (2.20), we see that $\Theta(\lambda)$ is J -coisometry for $\lambda \in i\mathbb{R}$. As J is a finite matrix we actually have (see [4]):

- for $\lambda \in i\mathbb{R}$, $\Theta(\lambda)$ is J -unitary:

$$J - \Theta(\lambda)^*J\Theta(\lambda) = J - \Theta(\lambda)J\Theta(\lambda)^* = 0 \quad \text{for } \lambda \in i\mathbb{R}. \quad (2.21)$$

The significance of the assumption that $\Gamma_{\mathfrak{D}}$ is not only invertible but also positive definite is that

- for $\lambda \in \Pi_+$ a point of analyticity for Θ , $\Theta(\lambda)$ is J -bicontractive:

$$J - \Theta(\lambda)^*J\Theta(\lambda) \succeq 0, \quad J - \Theta(\lambda)J\Theta(\lambda)^* \succeq 0 \quad \text{for } \lambda \in \Pi_+. \quad (2.22)$$

Here we make use of the fact that J -co-contractive is equivalent to J -contractive in the matrix case (see [4]). These last two observations have critical consequences. Again writing out Θ and J as

$$\Theta(\lambda) = \begin{bmatrix} \Theta_{11}(\lambda) & \Theta_{12}(\lambda) \\ \Theta_{21}(\lambda) & \Theta_{22}(\lambda) \end{bmatrix}, \quad J = \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix},$$

relations (2.21) and (2.22)) give us (with the variable λ suppressed)

$$\begin{bmatrix} \Theta_{11}\Theta_{11}^* - \Theta_{12}\Theta_{12}^* & \Theta_{11}\Theta_{21}^* - \Theta_{12}\Theta_{22}^* \\ \Theta_{21}\Theta_{11}^* - \Theta_{22}\Theta_{12}^* & \Theta_{21}\Theta_{21}^* - \Theta_{22}\Theta_{22}^* \end{bmatrix} \preceq \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix}$$

for λ a point of analyticity of Θ in Π_+ with equality for λ in $i\mathbb{R} = \partial\Pi_+$ (including the point at infinity). In particular,

$$\Theta_{21}\Theta_{21}^* - \Theta_{22}\Theta_{22}^* \preceq -I_m$$

or equivalently,

$$\Theta_{21}\Theta_{21}^* + I_m \preceq \Theta_{22}\Theta_{22}^*. \quad (2.23)$$

Hence, $\Theta_{22}(\lambda)$ is invertible at all points λ of analyticity in Π_+ , namely, $\Pi_+ \setminus \sigma(W)$, and then, since multiplying on the left by Θ_{22}^{-1} and on the right by its adjoint preserves the inequality, we get

$$\Theta_{22}^{-1}\Theta_{21}\Theta_{21}^*\Theta_{22}^{*-1} + \Theta_{22}^{-1}\Theta_{22}^{*-1} \preceq I_m. \quad (2.24)$$

We conclude:

- Θ_{22}^{-1} has analytic continuation to a contractive $m \times m$ -matrix function on all of Π_+ and $\Theta_{22}^{-1}\Theta_{21}$ has analytic continuation to an analytic $m \times p$ -matrix rational function which is pointwise strictly contractive on the closed right half plane $\overline{\Pi}_+ = \Pi_+ \cup i\mathbb{R}$.

It remains to make the connection of Θ with the BTOA-NP interpolation problem. Let us introduce some additional notation. For N a positive integer, $H_N^2(\Pi_+)$ is short-hand notation for the \mathbb{C}^N -valued Hardy space $H^2(\Pi_+) \otimes \mathbb{C}^N$ over the right half plane Π_+ . Similarly $L_N^2(i\mathbb{R}) = L^2(i\mathbb{R}) \otimes \mathbb{C}^N$ is the \mathbb{C}^N -valued L^2 -space over the imaginary line $i\mathbb{R}$.

It is well known (see e.g. [28]) that the space $H_N^2(\Pi_+)$ (consisting of analytic functions on Π_+) can be identified with a subspace of $L_N^2(i\mathbb{R})$ (consisting of measurable functions on $i\mathbb{R}$ defined only almost everywhere with respect to linear Lebesgue measure) via the process of taking nontangential limits from Π_+ to a point on $i\mathbb{R}$. Similarly the Hardy space $H_N^2(\Pi_-)$ over the left half plane can also be identified with a subspace (still denoted as $H_N^2(\Pi_-)$) of $L_N^2(i\mathbb{R})$, and, after these identifications, $H_N^2(\Pi_-) = H_N^2(\Pi_+)^{\perp}$ as subspaces of $L_N^2(i\mathbb{R})$:

$$L_N^2(i\mathbb{R}) = H_N^2(\Pi_+) \oplus H_N^2(\Pi_-).$$

We shall use these identifications freely in the discussion to follow. Given the Π_+ -admissible interpolation data set (1.14), we define a subspace of $L_{p+m}^2(i\mathbb{R})$ by

$$\begin{aligned} \mathcal{M}_{\mathfrak{D}} = & \left\{ \begin{bmatrix} V \\ U \end{bmatrix} (\lambda I - W)^{-1}x + \begin{bmatrix} f(\lambda) \\ g(\lambda) \end{bmatrix} : x \in \mathbb{C}^{n_W} \text{ and } \begin{bmatrix} f \\ g \end{bmatrix} \in H_{p+m}^2(\Pi_+) \right. \\ & \left. \text{such that } \sum_{z_0 \in \Pi_+} \text{Res}_{\lambda=z_0} (\lambda I - Z)^{-1} \begin{bmatrix} X & -Y \end{bmatrix} \begin{bmatrix} f(\lambda) \\ g(\lambda) \end{bmatrix} = \Gamma x \right\}. \end{aligned} \quad (2.25)$$

and a subspace of $L_m^2(i\mathbb{R})$ by

$$\mathcal{M}_{\mathfrak{D},-} = \{U(\lambda I - W)^{-1}x : x \in \mathbb{C}^{n_W}\} \oplus H_m^2(\Pi_+).$$

Using Π_+ -admissibility assumptions on the data set \mathfrak{D} one can show (we refer to [9] for details, subject to the disclaimer in Remark 2.1 below) that

$$\mathcal{M}_{\mathfrak{D},-} = P \begin{bmatrix} 0 \\ L_m^2(i\mathbb{R}) \end{bmatrix} \mathcal{M}_{\mathfrak{D}}.$$

Furthermore, a variant of the Beurling-Lax Theorem assures us that there is a $m \times m$ -matrix inner function ψ on Π_+ so that

$$\mathcal{M}_{\mathfrak{D},-} = \psi^{-1} \cdot H_m^2(\Pi_+). \quad (2.26)$$

Making use of [9, Theorem 6.1] applied to the null-pole triple $(U, W; \emptyset, \emptyset; \emptyset)$ over Π_+ , one can see that such a ψ (defined uniquely up to a constant unitary factor on the left) is given by the state-space realization formula

$$\psi(z) = I_m - UP^{-1}(zI + W^*)^{-1}U^*, \quad (2.27)$$

where the positive definite matrix P is uniquely defined from the Lyapunov equation $PW + W^*P = U^*U$, with ψ^{-1} given by

$$\psi(z)^{-1} = I_m + U(zI - W)^{-1}P^{-1}U^*, \quad (2.28)$$

i.e., that (U, W) is the right null pair of ψ . Furthermore, a second application of [9, Theorem 6.1] to the null-pole triple $(\begin{bmatrix} V \\ U \end{bmatrix}, W; Z, \begin{bmatrix} X & -Y \end{bmatrix}; \Gamma)$ over Π_+ leads to:

- $\mathcal{M}_{\mathfrak{D}}$ has the Beurling-Lax-type representation

$$\mathcal{M}_{\mathfrak{D}} = \Theta \cdot H_{p+m}^2(\Pi_+). \quad (2.29)$$

By projecting the identity (2.29) onto the bottom component and recalling the identity (2.26), we see that

$$\begin{bmatrix} \Theta_{21} & \Theta_{22} \end{bmatrix} H_{p+m}^2(\Pi_+) = \mathcal{M}_{\mathfrak{D},-} = \psi^{-1} H_m^2(\Pi_+). \quad (2.30)$$

On the other hand, for any $\begin{bmatrix} f_+ \\ f_- \end{bmatrix} \in H_{p+m}^2(\Pi_+)$, we have

$$\begin{aligned} \begin{bmatrix} \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} f_+ \\ f_- \end{bmatrix} &= \Theta_{21}f_+ + \Theta_{22}f_- \\ &= \Theta_{22}(\Theta_{22}^{-1}\Theta_{21}f_+ + f_-) \in \Theta_{22}H_m^2(\Pi_+), \end{aligned} \quad (2.31)$$

since $\Theta_{22}^{-1}\Theta_{21}$ is analytic on Π_+ . Since the reverse containment

$$\Theta_{22} \cdot H_m^2(\Pi_+) \subset \begin{bmatrix} \Theta_{21} & \Theta_{22} \end{bmatrix} \cdot H_{p+m}^2(\Pi_+)$$

is obvious, we may combine (2.30) and (2.31) to conclude that

$$\Theta_{22} \cdot H_m^2(\Pi_+) = \begin{bmatrix} \Theta_{21} & \Theta_{22} \end{bmatrix} \cdot H_{p+m}^2(\Pi_+) = \psi^{-1} \cdot H_m^2(\Pi_+). \quad (2.32)$$

It turns out that the geometry of $\mathcal{M}_{\mathfrak{D}}$ encodes the interpolation conditions:

- *An analytic function $S: \Pi_+ \rightarrow \mathbb{C}^{p \times m}$ satisfies the interpolation conditions (1.11), (1.12), (1.13) if and only if*

$$\begin{bmatrix} S \\ I_m \end{bmatrix} \cdot \mathcal{M}_{\mathfrak{D},-} \subset \mathcal{M}_{\mathfrak{D}}. \quad (2.33)$$

It remains to put the pieces together to arrive at the linear-fractional parametrization (1.20) for the set of all solutions (and thereby prove that solutions exist). Suppose that $S \in \mathcal{S}^{p \times m}(\Pi_+)$ satisfies the interpolation conditions (1.11), (1.12), (1.13). As a consequence of the criterion (2.33) combined with (2.26) and (2.29), we have

$$\begin{bmatrix} S \\ I_m \end{bmatrix} \psi^{-1} \cdot H_m^2(\Pi_+) \subset \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \cdot H_{p+m}^2(\Pi_+).$$

Hence there must be a $(p+m) \times m$ matrix function $\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \in H_{(p+m) \times m}^2(\Pi_+)$ so that

$$\begin{bmatrix} S \\ I_m \end{bmatrix} \psi^{-1} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}. \quad (2.34)$$

We next combine this identity with the J -unitary property of (2.21): for the (suppressed) argument $\lambda \in i\mathbb{R}$ we have

$$\begin{aligned} 0 &\succeq \psi^{-1*}(S^*S - I)\psi^{-1} = \psi^{-1*} \begin{bmatrix} Q_1^* & Q_2^* \end{bmatrix} \Theta^* J \Theta \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \psi^{-1} \\ &= \psi^{-1*} \begin{bmatrix} Q_1^* & Q_2^* \end{bmatrix} J \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \psi^{-1} \\ &= \psi^{-1*}(Q_1^*Q_1 - Q_2^*Q_2)\psi^{-1}. \end{aligned}$$

We conclude that

$$\|Q_1(\lambda)x\|^2 \leq \|Q_2(\lambda)x\|^2 \text{ for all } x \in \mathbb{C}^m \text{ and } \lambda \in i\mathbb{R}.$$

In particular, if $Q_2(\lambda)x = 0$ for some $\lambda \in i\mathbb{R}$ and $x \in \mathbb{C}^m$, then also $Q_1(\lambda)x = 0$ and hence

$$\psi(\lambda)^{-1}x = \begin{bmatrix} \Theta_{21}(\lambda) & \Theta_{22}(\lambda) \end{bmatrix} \begin{bmatrix} Q_1(\lambda) \\ Q_2(\lambda) \end{bmatrix} x = 0,$$

which forces $x = 0$ since ψ is rational matrix inner. We conclude:

- *for $\lambda \in i\mathbb{R}$, $Q_2(\lambda)$ is invertible and $G(\lambda) = Q_1(\lambda)Q_2(\lambda)^{-1}$ is a contraction.*

The next step is to apply a winding-number argument to get similar results for $\lambda \in \Pi_+$. From the bottom component of (2.34) we have, again for the moment with $\lambda \in i\mathbb{R}$,

$$\psi^{-1} = \Theta_{21}Q_1 + \Theta_{22}Q_2 = \Theta_{22}(\Theta_{22}^{-1}\Theta_{21}G + I_m)Q_2. \quad (2.35)$$

We conclude that, for the argument $\lambda \in i\mathbb{R}$,

$$\text{wno det}(\psi^{-1}) = \text{wno det}(\Theta_{22}) + \text{wno det}(\Theta_{22}^{-1}\Theta_{21}G + I_m) + \text{wno det}(Q_2) \quad (2.36)$$

where we use the notation $\text{wno } f$ to indicate *winding number* or *change of argument* of the function f as the variable runs along the imaginary line. Since both $\det \Theta_{22}^{-1}$ and $\det \psi$ are analytic on $\overline{\Pi}_+$, a consequence of the identity (2.32) is that

$$\text{wno } \det(\psi^{-1}) = \text{wno } \det(\Theta_{22}). \quad (2.37)$$

Combining the two last equalities gives

$$\text{wno } \det(\Theta_{22}^{-1} \Theta_{21} G + I_m) + \text{wno } \det(Q_2) = 0. \quad (2.38)$$

We have already observed that

$$\|\Theta_{22}(\lambda)^{-1} \Theta_{21}(\lambda)\| < 1 \quad \text{and} \quad \|G(\lambda)\| \leq 1 \quad \text{for } \lambda \in i\mathbb{R}.$$

Hence, for $0 \leq t \leq 1$ we have $\|t\Theta_{22}(\lambda)^{-1} \Theta_{21}(\lambda)G(\lambda)\| < 1$ and hence $t\Theta_{22}(\lambda)^{-1} \Theta_{21}(\lambda)G(\lambda) + I$ is invertible for $\lambda \in i\mathbb{R}$ for all $0 \leq t \leq 1$. Hence

$$i(t) := \text{wno } \det(t\Theta_{22}(\lambda)^{-1} \Theta_{21}(\lambda)G(\lambda) + I)$$

is well defined and independent of t for $0 \leq t \leq 1$. As clearly $i(0) = 0$, it follows that

$$i(1) = \text{wno } \det(\Theta_{22}(\lambda)^{-1} \Theta_{21}(\lambda)G(\lambda) + I) = 0$$

which, on account of (2.38), implies $\text{wno } \det(Q_2) = 0$. As Q_2 is analytic on Π_+ , we conclude that $\det Q_2$ has no zeros in Π_+ , i.e., Q_2^{-1} is analytic on Π_+ . By the maximum modulus theorem it then follows that $G(\lambda) := Q_1(\lambda)Q_2(\lambda)^{-1}$ is in the Schur class $\mathcal{S}^{p \times m}(\Pi_+)$. Furthermore, from (2.34) we have

$$\begin{bmatrix} S \\ I_m \end{bmatrix} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} G \\ I \end{bmatrix} Q_2 \psi. \quad (2.39)$$

From the bottom component we read off that $Q_2 \psi = (\Theta_{21}G + \Theta_{22})^{-1}$. From the first component we then get

$$S = (\Theta_{11}G + \Theta_{12})Q_2 \psi = (\Theta_{11}G + \Theta_{12})(\Theta_{21}G + \Theta_{22})^{-1}$$

and the representation (1.20) follows.

Conversely, if $G \in \mathcal{S}^{p \times m}(\Pi_+)$, we can reverse the above argument (with $Q_1(\lambda) = G(\lambda)$ and $Q_2(\lambda) = I_m$) to see that S of the form (1.20) is a Schur-class solution of the interpolation conditions (1.11), (1.12), (1.13). \square

Remark 2.1. The theory from [9] is worked out explicitly only with $H_m^2(\Pi_+)$ replaced by its rational subspace $\text{Rat } H_m^2$ consisting of elements of H_m^2 with rational-function column entries, and similarly $H_m^2(\Pi_-)$ and $L^2(i\mathbb{R})$ replaced by their respective rational subspaces $\text{Rat } H_m^2(\Pi_-)$ and $\text{Rat } L^2(i\mathbb{R})$. Nevertheless the theory is easily adapted to the L^2 -setting here. Subspaces \mathcal{M} of $L_{p+m}^2(i\mathbb{R})$ having a representation of the form (2.25) (with $\begin{bmatrix} U \\ V \end{bmatrix}, W, \begin{bmatrix} X & -Y \end{bmatrix}, Z, \Gamma$ all equal to finite matrices rather than infinite-dimensional operators) are characterized by the conditions: (1) \mathcal{M} is forward-shift invariant, i.e., \mathcal{M} is invariant under multiplication by the function

$\chi(\lambda) = \frac{\lambda-1}{\lambda+1}$, (2) the subspace $(\mathcal{M} + H_{p+m}^2(\Pi_+))/H_{p+m}^2(\Pi_+)$ has finite dimension, and (3) the quotient space $\mathcal{M}/(\mathcal{M} \cap H_{p+m}^2(\Pi_+))$ has finite dimension. The representation (2.26) with ψ^{-1} of the form (2.28) with finite matrices U, W, P is roughly the special case of the statement above where $\mathcal{M} = \mathcal{M}_{\mathfrak{D},-} \supset H_m^2(\Pi_+)$. The analogue of such representations (2.25) and (2.26)–(2.28) for more general full-range pure forward shift-invariant subspaces of $L_{p+m}^2(\Pi_+)$ (or dually of full-range pure backward shift-invariant subspaces of $L_{p+m}^2(\Pi_+)$) involving infinite-dimensional (even unbounded) operators $\begin{bmatrix} U \\ V \end{bmatrix}, W, Z, \begin{bmatrix} X & -Y \end{bmatrix}, \Gamma$ is worked out in the Virginia Tech dissertation of Austin Amaya [3].

2.1. The Fundamental Matrix Inequality approach of Potapov. The linear fractional parametrization formula (1.20) can be alternatively established by the Potapov's method of the Fundamental Matrix Inequalities. As we will see, this method bypasses the winding number argument.

Consider a Π_+ -admissible BTOA interpolation data set \mathfrak{D} as in (1.14) giving rise to the collection (1.11), (1.12), (1.13) of BTOA interpolation conditions imposed on a Schur-class function $S^{p \times m}(\Pi_+)$. We assume that $\Gamma_{\mathfrak{D}}$ is positive definite. We form the matrix $\Theta(\lambda)$ as in (2.17)–(2.18) and assume all knowledge of all the properties of Θ falling out of the positive-definiteness of $\Gamma_{\mathfrak{D}}$, specifically (2.19)–(2.23) above.

The main idea is to extend the interpolation data by one extra interpolation node $z \in \Pi_+$ with the corresponding full-range value $S(z)$, i.e., by the tautological full-range interpolation condition

$$S(z) = S(z) \quad (2.40)$$

where z is a generic point in the right half plane. To set up this augmented problem as a BTOA problem, we have a choice as to how we incorporate the global generic interpolation condition (2.40) into the BTOA formalism: (a) as a LTOA interpolation condition:

$$(X_z S)^{\wedge L}(Z_z) = Y_z \quad \text{where} \quad X_z = I_p, Y_z = S(z), Z_z = zI_p, \quad (2.41)$$

or as a RTOA interpolation condition:

$$(S U_z)^{\wedge R}(W_z) = V_z \quad \text{where} \quad U_z = I_m, V_z = S(z), W_z = zI_m. \quad (2.42)$$

We choose here to work with the left versions (2.41) exclusively; working with the right version (2.42) will give seemingly different but in the end equivalent parallel results.

As a first step, we wish to combine (1.11) and (2.41) into a single LTOA interpolation condition. This is achieved by augmenting the matrices (Z, X, Y) to the augmented triple $(Z_{\text{aug}}, X_{\text{aug}}, Y_{\text{aug}})$ given by

$$Z_{\text{aug}} = \begin{bmatrix} Z & 0 \\ 0 & zI_p \end{bmatrix}, \quad X_{\text{aug}} = \begin{bmatrix} X \\ I_p \end{bmatrix}, \quad Y_{\text{aug}} = \begin{bmatrix} Y \\ S(z) \end{bmatrix}.$$

Here all matrices indexed by aug depend on the parameter z , but for the moment we suppress this dependence from the notation. As the RTOA-interpolation conditions for the augmented problem remain the same as in the original problem (namely, (1.12)), we set

$$U_{\text{aug}} = U, \quad V_{\text{aug}} = V, \quad W_{\text{aug}} = W.$$

We therefore take the **augmented data set** $\mathfrak{D}_{\text{aug}}$ to have the form

$$\mathfrak{D}_{\text{aug}} = (X_{\text{aug}}, Y_{\text{aug}}, Z_{\text{aug}}; U_{\text{aug}}, V_{\text{aug}}, W_{\text{aug}}; \Gamma_{\text{aug}}) \quad (2.43)$$

where the coupling matrix Γ_{aug} is still to be determined.

We know that Γ_{aug} must solve the Sylvester equation (1.15) associated with the data set $\mathfrak{D}_{\text{aug}}$, i.e., Γ_{aug} must have the form $\Gamma_{\text{aug}} = \begin{bmatrix} \Gamma_{\text{aug},1} \\ \Gamma_{\text{aug},2} \end{bmatrix}$ with

$$\begin{bmatrix} \Gamma_{\text{aug},1} \\ \Gamma_{\text{aug},2} \end{bmatrix} W - \begin{bmatrix} Z & 0 \\ 0 & zI_p \end{bmatrix} \begin{bmatrix} \Gamma_{\text{aug},1} \\ \Gamma_{\text{aug},2} \end{bmatrix} = \begin{bmatrix} X \\ I_p \end{bmatrix} V - \begin{bmatrix} Y \\ S(z) \end{bmatrix} U.$$

Equivalently, $\Gamma_{\text{aug}} = \begin{bmatrix} \Gamma_{\text{aug},1} \\ \Gamma_{\text{aug},2} \end{bmatrix}$ is determined by the decoupled system of equations

$$\begin{aligned} \Gamma_{\text{aug},1} W - Z \Gamma_{\text{aug},1} &= X V - Y U, \\ \Gamma_{\text{aug},2} W - (z I_p) \Gamma_{\text{aug},2} &= V - S(z) U. \end{aligned} \quad (2.44)$$

In addition, the third augmented interpolation condition takes the form

$$\left(\begin{bmatrix} X \\ I_p \end{bmatrix} S U \right)^{\wedge L,R} \left(\begin{bmatrix} Z & 0 \\ 0 & z I_p \end{bmatrix}, W \right) = \begin{bmatrix} \Gamma_{\text{aug},1} \\ \Gamma_{\text{aug},2} \end{bmatrix}$$

which can be decoupled into two independent bitangential interpolation conditions

$$(X S U)^{\wedge L,R}(Z, W) = \Gamma_{\text{aug},1}, \quad (I_p S U)^{\wedge L,R}(z I_p, W) = \Gamma_{\text{aug},2}. \quad (2.45)$$

From the first of the conditions (2.45) coupled with the interpolation condition (1.13), we are forced to take $\Gamma_{\text{aug},1} = \Gamma$.

Since the point $z \in \Pi_+$ is generic, we may assume as a first case that z is disjoint from the spectrum $\sigma(W)$ of W . Then we can solve the second of the equations (2.44) uniquely for $\Gamma_{\text{aug},2}$:

$$\Gamma_{\text{aug},2} = (S(z)U - V)(zI_{n_W} - W)^{-1}. \quad (2.46)$$

A consequence of the RTOA interpolation condition (1.12) is that the right-hand side of (2.46) has analytic continuation to all of Π_+ . It is not difficult to see that $(I_p S U)^{\wedge L,R}(z I_p, W)$ in general is just the value of this analytic continuation at the point z ; we conclude that the formula (2.46) holds also at points z in $\sigma(W)$ with proper interpretation. In this way we have completed the computation of the augmented data set (2.43):

$$\mathfrak{D}(z) := \mathfrak{D}_{\text{aug}} = \left(\begin{bmatrix} Z & 0 \\ 0 & z I_p \end{bmatrix}, \begin{bmatrix} X \\ I_p \end{bmatrix}, \begin{bmatrix} Y \\ S(z) \end{bmatrix}; U, V, W; \begin{bmatrix} \Gamma \\ T_{S,1}(z) \end{bmatrix} \right). \quad (2.47)$$

where we set

$$T_{S,1}(z) = (S(z)U - V)(zI_{n_W} - W)^{-1}. \quad (2.48)$$

We next compute the Pick matrix $\mathbf{\Gamma}_{\mathfrak{D}_{\text{aug}}(z)}$ for the augmented data set $\mathfrak{D}_{\text{aug}}$ (2.47) according to the recipe (1.16)–(1.18). Thus

$$\mathbf{\Gamma}_{\mathfrak{D}_{\text{aug}}(z)} = \begin{bmatrix} \Gamma_{\text{aug},L} & \Gamma_{\text{aug}} \\ (\Gamma_{\text{aug}})^* & \Gamma_{\text{aug},R} \end{bmatrix}, \quad \text{where} \quad \Gamma_{\text{aug}} = \begin{bmatrix} \Gamma \\ T_{S,1}(z) \end{bmatrix}, \quad \Gamma_{\text{aug},R} = \Gamma_R,$$

and where $\Gamma_{\text{aug},L} = \begin{bmatrix} \Gamma_{\text{aug},L11} & \Gamma_{\text{aug},L12} \\ \Gamma_{\text{aug},L21} & \Gamma_{\text{aug},L22} \end{bmatrix}$ is determined by the Lyapunov equation (1.16) adapted to the interpolation data set $\mathfrak{D}(z)$:

$$\begin{aligned} & \begin{bmatrix} \Gamma_{\text{aug},L11} & \Gamma_{\text{aug},L12} \\ \Gamma_{\text{aug},L21} & \Gamma_{\text{aug},L22} \end{bmatrix} \begin{bmatrix} Z^* & 0 \\ 0 & \bar{z}I_p \end{bmatrix} + \begin{bmatrix} Z & 0 \\ 0 & zI_p \end{bmatrix} \begin{bmatrix} \Gamma_{\text{aug},L11} & \Gamma_{\text{aug},L12} \\ \Gamma_{\text{aug},L21} & \Gamma_{\text{aug},L22} \end{bmatrix} \\ &= \begin{bmatrix} X \\ I_p \end{bmatrix} \begin{bmatrix} X^* & I_p \end{bmatrix} - \begin{bmatrix} Y \\ S(z) \end{bmatrix} \begin{bmatrix} Y^* & S(z)^* \end{bmatrix}. \end{aligned}$$

One can solve this equation uniquely for $\Gamma_{\text{aug},Lij}$ ($i, j = 1, 2$) with the result

$$\begin{aligned} \Gamma_{\text{aug},L11} &= \Gamma_L, \quad \Gamma_{\text{aug},L21} = (\Gamma_{\text{aug},L12})^* = T_{S,2}(z), \\ \Gamma_{\text{aug},L22} &= \frac{I - S(z)S(z)^*}{z + \bar{z}} \end{aligned}$$

where we set

$$T_{S,2}(z) := (X^* - S(z)Y^*)(zI_{n_Z} + Z^*)^{-1}. \quad (2.49)$$

In this way we arrive at the Pick matrix for data set $\mathfrak{D}(z)$, denoted for convenience as $\mathbf{\Gamma}_{\mathfrak{D}}(z)$ rather than as $\mathbf{\Gamma}_{\mathfrak{D}(z)}$:

$$\mathbf{\Gamma}_{\mathfrak{D}}(z) = \begin{bmatrix} \Gamma_L & T_{S,2}(z)^* & \Gamma \\ T_{S,2}(z) & \frac{I - S(z)S(z)^*}{z + \bar{z}} & T_{S,1}(z) \\ \Gamma^* & T_{S,1}(z)^* & \Gamma_R \end{bmatrix}.$$

If we interchange the second and third rows and then also the second and third columns (i.e., conjugate by a permutation matrix), we get a new matrix having the same inertia; for simplicity from now on we use the same notation $\mathbf{\Gamma}_{\mathfrak{D}}(z)$ for this transformed matrix:

$$\mathbf{\Gamma}_{\mathfrak{D}}(z) = \begin{bmatrix} \Gamma_L & \Gamma & T_{S,2}(z)^* \\ \Gamma^* & \Gamma_R & T_{S,1}(z)^* \\ T_{S,2}(z) & T_{S,1}(z) & \frac{I - S(z)S(z)^*}{z + \bar{z}} \end{bmatrix}.$$

Had we started with a finite number $\mathbf{z} = \{z_1, \dots, z_N\}$ of generic interpolation nodes in Π_+ rather than a single generic point z and augmented the interpolation conditions (1.11), (1.12), (1.13) with the collection of tautological interpolation conditions

$$S(z_i) = S(z_i) \quad \text{for } i = 1, \dots, N$$

modeled as the additional LTOA interpolation condition

$$(X_{\mathbf{z}}S)^{\wedge L}(Z_{\mathbf{z}}) = Y_{\mathbf{z}}$$

where

$$Z_{\mathbf{z}} = \begin{bmatrix} z_1 I_p & & 0 \\ & \ddots & \\ 0 & & z_N I_p \end{bmatrix}, \quad X_{\mathbf{z}} = \begin{bmatrix} I_p \\ \vdots \\ I_p \end{bmatrix}, \quad Y_{\mathbf{z}} = \begin{bmatrix} S(z_1) \\ \vdots \\ S(z_N) \end{bmatrix},$$

the same analysis as above would lead us to the following conclusion: *there is a matrix function S in the Schur class $\mathcal{S}^{p \times m}(\Pi_+)$ satisfying the interpolation conditions (1.11), (1.12), (1.13) if and only if, for $\mathbf{z} = \{z_1, \dots, z_N\}$ any collection of N distinct points in Π_+ , the associated augmented Pick matrix $\mathbf{\Gamma}_{\mathfrak{D}}(\mathbf{z})$ is positive-semidefinite, where*

$$\mathbf{\Gamma}_{\mathfrak{D}}(\mathbf{z}) = \begin{bmatrix} \Gamma_L & \Gamma & T_{S,2}(z_1)^* & \cdots & T_{S,2}(z_N)^* \\ \Gamma^* & \Gamma_R & T_{S,1}(z_1)^* & \cdots & T_{S,1}(z_N)^* \\ T_{S,2}(z_1) & T_{S,1}(z_1) & \frac{I - S(z_1)S(z_1)^*}{z_1 + \bar{z}_1} & \cdots & \frac{I - S(z_1)S(z_N)^*}{z_1 + \bar{z}_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{S,2}(z_N) & T_{S,1}(z_N) & \frac{I - S(z_N)S(z_1)^*}{z_1 + \bar{z}_N} & \cdots & \frac{I - S(z_N)S(z_N)^*}{z_N + \bar{z}_N} \end{bmatrix} \succeq 0.$$

As the finite set of points $\mathbf{z} = \{z_1, \dots, z_N\}$ ($N = 1, 2, \dots$) is an arbitrary finite subset of Π_+ , this condition in turn amounts to the assertion that the kernel $\mathbf{\Gamma}_{\mathfrak{D}}(z, \zeta)$ defined by

$$\mathbf{\Gamma}_{\mathfrak{D}}(z, \zeta) = \begin{bmatrix} \Gamma_L & \Gamma & T_{S,2}(\zeta)^* \\ \Gamma^* & \Gamma_R & T_{S,1}(\zeta)^* \\ T_{S,2}(z) & T_{S,1}(z) & \frac{I - S(z)S(\zeta)^*}{z + \bar{\zeta}} \end{bmatrix} \quad (2.50)$$

is a positive kernel on Π_+ (see (2.1)). Observe from (2.49), (2.48) that

$$\begin{aligned} [T_{S,2}(z) \quad T_{S,1}(z)] &= -[I_p \quad -S(z)] \begin{bmatrix} -X^* & V \\ -Y^* & U \end{bmatrix} \begin{bmatrix} (zI + Z^*)^{-1} & 0 \\ 0 & (zI - W)^{-1} \end{bmatrix} \\ &= -[I_p \quad -S(z)] \mathbf{C}(zI - \mathbf{A})^{-1}, \end{aligned}$$

where \mathbf{C} and \mathbf{A} are defined as in (2.18). Taking the latter formula into account, we may write (2.50) in a more structured form as

$$\mathbf{\Gamma}_{\mathfrak{D}}(z, \zeta) = \begin{bmatrix} \Gamma_{\mathfrak{D}} & -(\bar{\zeta}I - \mathbf{A}^*)^{-1} \mathbf{C}^* \begin{bmatrix} I_p \\ -S(\zeta)^* \end{bmatrix} \\ -[I_p \quad -S(z)] \mathbf{C}(zI - \mathbf{A})^{-1} & \frac{I - S(z)S(\zeta)^*}{z + \bar{\zeta}} \end{bmatrix}. \quad (2.51)$$

Since the matrix $\mathbf{\Gamma}_{\mathfrak{D}}$ is positive definite, the kernel (2.51) is positive if and only if the Schur complement of $\Gamma_{\mathfrak{D}}$ is a positive kernel on $\Pi_+ \setminus \sigma(W)$ and therefore, admits a unique positive extension to the whole Π_+ :

$$\frac{I - S(z)S(\zeta)^*}{z + \bar{\zeta}} - [I_p \quad -S(z)] \mathbf{C}(zI - \mathbf{A})^{-1} \Gamma_{\mathfrak{D}}^{-1} (\bar{\zeta}I - \mathbf{A}^*)^{-1} \mathbf{C}^* \begin{bmatrix} I_p \\ -S(\zeta)^* \end{bmatrix} \succeq 0.$$

The latter can be written as

$$[I_p \quad -S(z)] \left\{ \frac{J}{z + \bar{\zeta}} - \mathbf{C}(zI - \mathbf{A})^{-1} \Gamma_{\mathfrak{D}}^{-1} (\bar{\zeta}I - \mathbf{A}^*)^{-1} \mathbf{C}^* \right\} \begin{bmatrix} I_p \\ -S(\zeta)^* \end{bmatrix} \succeq 0,$$

and finally, upon making use of (2.20), as

$$[I_p \quad -S(z)] \frac{\Theta(z)J\Theta(\zeta)^*}{z + \bar{\zeta}} \begin{bmatrix} I_p \\ -S(\zeta)^* \end{bmatrix} \succeq 0. \quad (2.52)$$

We next define two functions Q_1 and Q_2 by the formula

$$\begin{bmatrix} Q_2(z) & -Q_1(z) \end{bmatrix} = [I_p \quad -S(z)] \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix}, \quad (2.53)$$

and write (2.52) in terms of these functions as

$$\begin{bmatrix} Q_2(z) & -Q_1(z) \end{bmatrix} \frac{J}{z + \bar{\zeta}} \begin{bmatrix} Q_2(\zeta)^* \\ -Q_1(\zeta)^* \end{bmatrix} = \frac{Q_2(z)Q_2(\zeta)^* - Q_1(z)Q_1(\zeta)^*}{z + \bar{\zeta}} \succeq 0.$$

By Leech's theorem [35], there exists a Schur-class function $G \in \mathcal{S}^{p \times m}(\Pi_+)$ such that

$$Q_2 G = Q_1,$$

which, in view of (2.53) can be written as

$$(\Theta_{11} - S\Theta_{21})G = S\Theta_{22} - \Theta_{12},$$

or equivalently, as

$$S(\Theta_{21}G + \Theta_{22}) = \Theta_{11}G + \Theta_{12}. \quad (2.54)$$

Note that $\Theta_{22}(z)$ is invertible and that $\Theta_{22}(z)^{-1}\Theta_{21}(z)$ is strictly contractive on all of $\Pi_+ \setminus \sigma(W)$ (and then on all of Π_+ by analytic continuation) as a consequence of the bullet immediately after (2.24) above. As G is in the Schur class and hence is contractive on all of Π_+ , it follows that $\Theta_{22}(z)^{-1}\Theta_{21}(z)G(z) + I_m$ is invertible on all of Π_+ . Hence

$$\Theta_{21}(z)G(z) + \Theta_{22}(z) = \Theta_{22}(z)(\Theta_{22}(z)^{-1}\Theta_{21}(z)G(z) + I_m)$$

is invertible for all $z \in \Pi_+$ and we can solve (2.54) for S arriving at the formula (1.20).

Remark 2.2. Note that in this Potapov approach to the derivation of the linear-fractional parametrization via the Fundamental Matrix Inequality, the winding-number argument appearing in the state-space approach never appears. What apparently replaces it, once everything is properly organized, is the theorem of Leech.

2.2. Positive kernels and reproducing kernel Hilbert spaces. Assume now that we are given a Π_+ -admissible interpolation data set and that the Pick matrix $\Gamma_{\mathfrak{D}}$ is invertible. Then one can define the matrix function $\Theta(z)$ as in (2.17) and then $K_{\Theta,J}(z, \zeta) = \frac{J - \Theta(z)J\Theta(\zeta)^*}{z + \bar{\zeta}}$ is given by (2.20). A straightforward computation then shows that, for any $N = 1, 2, \dots$ with

points z_1, \dots, z_N in $\Pi_+ \setminus \sigma(W)$ and vectors $\mathbf{y}_1, \dots, \mathbf{y}_N$ in \mathbb{C}^{p+m} , we have

$$\begin{aligned} \sum_{i,j=1}^N \langle K_{\Theta,J}(z_i, z_j) \mathbf{y}_j, \mathbf{y}_i \rangle_{\mathbb{C}^{p+m}} &= \\ &= \sum_{i,j=1}^N \left\langle \Gamma_{\mathfrak{D}}^{-1} \left(\sum_{i=1}^N (\bar{z}_i I - \mathbf{A}^*)^{-1} \mathbf{C}_i^* \right), \sum_{j=1}^N (\bar{z}_j I - \mathbf{A}^*)^{-1} \mathbf{C}_j^* \right\rangle, \end{aligned}$$

and hence $K_{\Theta,J}$ is a positive kernel on $\Pi_+ \setminus \sigma(W)$ if $\Gamma_{\mathfrak{D}} \succ 0$. More generally, if $\Gamma_{\mathfrak{D}}$ has some number κ of negative eigenvalues, then for any choice of points $z_1, \dots, z_N \in \Pi_+ \setminus \sigma(W)$ the block Hermitian matrix

$$[K_{\Theta,J}(z_i, z_j)]_{i,j=1,\dots,N} \quad (2.55)$$

has at most κ negative eigenvalues. If we impose the controllability and observability assumptions on the matrix pairs (U, W) and (Z, X) , then there exist a choice of $z_1, \dots, z_N \in \Pi_+ \setminus \sigma(W)$ so that the matrix (2.55) has exactly κ negative eigenvalues, in which case we say that Θ is in the generalized J -Schur class $\mathcal{S}_{J,\kappa}(\Pi_+)$ (compare with the Kreĭn-Langer generalized Schur class discussed at the beginning of Section 5 below). In the case where $\Theta \in \mathcal{S}_{J,\kappa}(\Pi_+)$ with $\kappa > 0$, there is still associated a space of functions $\mathcal{H}(K_{\Theta,J})$ as in (2.2)–(2.3); the space $\mathcal{H}(K_{\Theta,J})$ is now a *Pontryagin space* with negative index equal to κ (see Section 3.1 for background on Pontryagin and Kreĭn spaces). In any case, in this way we arrive at yet another interpretation of the condition that $\Gamma_{\mathfrak{D}}$ be positive definite.

Theorem 2.3. *Assume that we are given a Π_+ -admissible interpolation data set with $\Gamma_{\mathfrak{D}}$ is invertible (so Θ and $K_{\Theta,J}$ are defined). Then $\mathcal{H}(K_{\Theta,J})$ is a Hilbert space if and only if $\Gamma_{\mathfrak{D}} \succ 0$.*

In Section 4 below (see display (4.26)) we shall spell this criterion out in more detail and arrive at another condition equivalent to positive-definiteness of the Pick matrix $\Gamma_{\mathfrak{D}}$.

3. THE GRASSMANNIAN/KREĬN-SPACE-GEOMETRY APPROACH TO THE BTOA-NP INTERPOLATION PROBLEM

In this section we sketch the Grassmannian/Kreĭn-space geometry proof of Theorem 1.2 based on the work in [10]—see also [5] for a more expository account and [8] for a more recent overview which also highlights the method in various multivariable settings. These treatments work with the Sarason [38] or Model-Matching [25] formulation of the Nevanlinna-Pick interpolation problem, while we work with the LTOA-interpolation formulation. The translation between the two is given in [9, Chapter 16] (where the Sarason/Model Matching formulation is called *divisor-remainder form*).

3.1. Kreĭn-space preliminaries. Let us first review a few preliminaries concerning Kreĭn spaces. A Kreĭn space by definition is a linear space \mathcal{K} endowed with an indefinite inner product $[\cdot, \cdot]$ which is *complete* in the following sense: there are two subspaces \mathcal{K}_+ and \mathcal{K}_- of \mathcal{K} such that the restriction of $[\cdot, \cdot]$ to $\mathcal{K}_+ \times \mathcal{K}_+$ makes \mathcal{K}_+ a Hilbert space while the restriction of $-\![\cdot, \cdot]$ to $\mathcal{K}_- \times \mathcal{K}_-$ makes \mathcal{K}_- a Hilbert space, and

$$\mathcal{K} = \mathcal{K}_+ [+] \mathcal{K}_- \quad (3.1)$$

is a $[\cdot, \cdot]$ -orthogonal direct sum decomposition of \mathcal{K} . In this case the decomposition (3.1) is said to form a *fundamental decomposition* for \mathcal{K} . Fundamental decompositions are never unique except in the trivial case where one of \mathcal{K}_+ or \mathcal{K}_- is equal to the zero space. If $\min(\dim \mathcal{K}_+, \dim \mathcal{K}_-) = \kappa < \infty$, then \mathcal{K} is called a Pontryagin space of index κ .

Unlike the case of Hilbert spaces where closed subspaces all look the same, there is a rich geometry for subspaces of a Kreĭn space. A subspace \mathcal{M} of a Kreĭn space \mathcal{K} is said to be *positive*, *isotropic*, or *negative* depending on whether $[u, u] \geq 0$ for all $u \in \mathcal{M}$, $[u, u] = 0$ for all $u \in \mathcal{M}$ (in which case it follows that $[u, v] = 0$ for all $u, v \in \mathcal{M}$ as a consequence of the Cauchy-Schwarz inequality), or $[u, u] \leq 0$ for all $u \in \mathcal{M}$. Given any subspace \mathcal{M} , we define the Kreĭn-space orthogonal complement $\mathcal{M}^{[\perp]}$ to consist of all $v \in \mathcal{K}$ such that $[u, v] = 0$ for all $u \in \mathcal{M}$. Note that the statement that \mathcal{M} is isotropic is just the statement that $\mathcal{M} \subset \mathcal{M}^{[\perp]}$. If it happens that $\mathcal{M} = \mathcal{M}^{[\perp]}$, we say that \mathcal{M} is a *Lagrangian* subspace of \mathcal{K} . Simple examples show that in general, unlike the Hilbert space case, it can happen that \mathcal{M} is a closed subspace of the Kreĭn space \mathcal{K} yet the space \mathcal{K} cannot be split at the \mathcal{K} -orthogonal direct sum of \mathcal{M} and $\mathcal{M}^{[\perp]}$ (e.g., this happens dramatically if \mathcal{M} is an isotropic subspace of \mathcal{K}). If \mathcal{M} is a subspace of \mathcal{K} for which this does happen, i.e., such that $\mathcal{K} = \mathcal{M} [+] \mathcal{M}^{[\perp]}$, we say that \mathcal{M} is a *regular subspace* of \mathcal{K} .

Examples of such subspaces arise from placing appropriate Kreĭn-space inner products on the direct sum $\mathcal{H}' \oplus \mathcal{H}$ of two Hilbert spaces and looking at graphs of operators of an appropriate class.

Example 3.1. Suppose that \mathcal{H}' and \mathcal{H} are two Hilbert spaces and we take \mathcal{K} to be the external direct sum $\mathcal{H}' \oplus \mathcal{H}$ with inner product

$$\left[\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix} \right] = \left\langle \begin{bmatrix} I_{\mathcal{H}'} & 0 \\ 0 & -I_{\mathcal{H}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix} \right\rangle_{\mathcal{H}' \oplus \mathcal{H}}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}' \oplus \mathcal{H}}$ is the standard Hilbert-space inner product on the direct-sum Hilbert space $\mathcal{H}' \oplus \mathcal{H}$. In this case it is easy to find a fundamental decomposition: take $\mathcal{K}_+ = \left[\begin{smallmatrix} \mathcal{H} \\ \{0\} \end{smallmatrix} \right]$ and $\mathcal{K}_- = \left[\begin{smallmatrix} \{0\} \\ \mathcal{H}' \end{smallmatrix} \right]$. Now let T be a bounded linear operator from \mathcal{H} to \mathcal{H}' and let \mathcal{M} be the graph of T :

$$\mathcal{M} = \mathcal{G}_T = \left\{ \begin{bmatrix} Tx \\ x \end{bmatrix} : x \in \mathcal{H} \right\} \subset \mathcal{K}.$$

Then a nice exercise is to work out the following facts:

- \mathcal{G}_T is negative if and only if $\|T\| \leq 1$, in which case \mathcal{G}_T is *maximal negative*, i.e., the subspace \mathcal{G}_T is not contained in any strictly larger negative subspace.
- \mathcal{G}_T is isotropic if and only if T is isometric ($T^*T = I_{\mathcal{H}}$).
- \mathcal{G}_T is Lagrangian if and only if T is unitary: $T^*T = I_{\mathcal{H}}$ and $TT^* = I_{\mathcal{H}'}$.

Let \mathcal{M} be a fixed subspace of a Kreĭn space \mathcal{K} and \mathcal{G} a closed subspace of \mathcal{M} . In order that \mathcal{G} be maximal negative as a subspace of \mathcal{K} , it is clearly necessary that \mathcal{G} be maximal negative as a subspace of \mathcal{M} . The following lemma (see [10] or [5] for the proof) identifies when the converse holds.

Lemma 3.2. *Suppose that \mathcal{M} is a closed subspace of a Kreĭn-space \mathcal{K} and \mathcal{G} is a negative subspace of \mathcal{M} . Then a subspace $\mathcal{G} \subset \mathcal{M}$ which is maximal-negative as a subspace of \mathcal{M} is automatically also maximal negative as a subspace of \mathcal{K} if and only if the Kreĭn-space orthogonal complement*

$$\mathcal{K}[-]\mathcal{M} = \{k \in \mathcal{K}: [k, m]_{\mathcal{K}} = 0 \text{ for all } m \in \mathcal{M}\}$$

is a positive subspace of \mathcal{K} .

3.2. The Grassmannian/Kreĭn-space approach to interpolation. Suppose now that we are given a Π_+ -admissible BTOA-interpolation data set as in (1.14). Let $\mathcal{M}_{\mathfrak{D}} \subset L_{p+m}^2(i\mathbb{R})$ be as in (2.25). We view $\mathcal{M}_{\mathfrak{D}}$ as a subspace of the Kreĭn space

$$\mathcal{K} = \begin{bmatrix} L_p^2(i\mathbb{R}) \\ \mathcal{M}_{\mathfrak{D},-} \end{bmatrix} = \begin{bmatrix} L_p^2(i\mathbb{R}) \\ \psi^{-1}H_m^2(\Pi_+) \end{bmatrix} \quad (3.2)$$

(where we use the notation in (2.26)) with Kreĭn-space inner product $[\cdot, \cdot]_J$ induced by the matrix $J = \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix}$:

$$\left[\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right]_J := \langle f_1, g_1 \rangle_{L_p^2(i\mathbb{R})} - \langle f_2, g_2 \rangle_{\psi^{-1}H_m^2(\Pi_+)}.$$

A key subspace in the Kreĭn-space geometry approach to the BTOA-NP problem is the J -orthogonal complement of $\mathcal{M}_{\mathfrak{D}}$ inside \mathcal{K} :

$$\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]} := \mathcal{K}[-]_J \mathcal{M}_{\mathfrak{D}} = \{f \in \mathcal{K}: [f, g]_J = 0 \text{ for all } g \in \mathcal{M}_{\mathfrak{D}}\}. \quad (3.3)$$

We then have the following result.

Theorem 3.3. *The BTOA-NP has a solution $S \in \mathcal{S}^{p \times m}(\Pi_+)$ if and only if the subspace $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$ (3.3) is a positive subspace of \mathcal{K} (3.2), i.e.,*

$$[f, f]_J \geq 0 \text{ for all } f \in \mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}. \quad (3.4)$$

If it is the case that $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$ is a Hilbert space in the \mathcal{K} -inner product, then there is rational J -inner function Θ so that

- (1) Θ provides a Beurling-Lax representation for $\mathcal{M}_{\mathfrak{D}}$ (2.29), and

- (2) *the set of all Schur-class solutions $S \in \mathcal{S}^{p \times m}(\Pi_+)$ of the interpolation conditions (1.11), (1.12), (1.13) is given by the linear-fractional parametrization formula (1.20) with $G \in \mathcal{S}^{p \times m}(\Pi_+)$.*

Sketch of the proof of Theorem 3.3. We first argue the $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$ being a positive subspace of \mathcal{K} is necessary for the BTOA-NP to have a solution. Let $S \in \mathcal{S}^{p \times m}(\Pi_+)$ be such a solution and let $M_S: \psi^{-1}H_m^2(\Pi_+) \rightarrow L_p^2(i\mathbb{R})$ be the operator of multiplication by S :

$$M_S: \psi^{-1}h \mapsto S \cdot \psi^{-1}h.$$

The operator norm of M_S is the same as the supremum norm of S over $i\mathbb{R}$:

$$\|M_S\|_{\text{op}} = \|S\|_{\infty} := \sup_{\lambda \in i\mathbb{R}} \|S(\lambda)\|.$$

Let us consider the graph space of M_S , namely

$$\mathcal{G}_S = \begin{bmatrix} M_S \\ I_m \end{bmatrix} \psi^{-1}H_m^2(\Pi_+) = \begin{bmatrix} S \\ I_m \end{bmatrix} \cdot \psi^{-1}H_m^2(\Pi_+). \quad (3.4)$$

By the first bullet in Example 3.1, it follows that

- $\|S\|_{\infty} \leq 1$ if and only if \mathcal{G}_S is a maximal negative subspace of \mathcal{K} .

Moreover, as a consequence of the criterion (2.33) for S to satisfy the interpolation conditions, we have

- S satisfies the interpolation conditions if and only if $\mathcal{G}_S \subset \mathcal{M}_{\mathfrak{D}}$.

By combining these two observations, we see that if S is a solution to the BTOA-NP, then the subspace \mathcal{G}_S is contained in $\mathcal{M}_{\mathfrak{D}}$ and is maximal negative in \mathcal{K} . It follows that $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$ is a positive subspace in \mathcal{K} as a consequence of Lemma 3.2. This verifies the necessity part in Theorem 3.3.

Conversely, suppose that \mathfrak{D} is a Π_+ -admissible BTOA-interpolation data set. Then we can form the space

$$\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]} \subset \mathcal{K} = \left[\begin{array}{c} L_p^2(i\mathbb{R}) \\ \psi^{-1}H_m^2(\Pi_+) \end{array} \right].$$

Suppose that $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$ is a positive subspace of \mathcal{K} . By Lemma 3.2, a subspace \mathcal{G} of $\mathcal{M}_{\mathfrak{D}}$ which is maximal negative as a subspace of $\mathcal{M}_{\mathfrak{D}}$ is also maximal negative as a subspace of \mathcal{K} . We also saw in the necessity argument that if the subspace \mathcal{G} has the form \mathcal{G}_S (3.4) for a matrix function S and $\mathcal{G}_S \subset \mathcal{M}_{\mathfrak{D}}$, then S satisfies the interpolation conditions (1.11), (1.12), (1.13). However, not all maximal negative subspaces $\mathcal{G} = \begin{bmatrix} T \\ I \end{bmatrix} \psi^{-1}H_m^2(\Pi_+)$ of \mathcal{K} have the form $\mathcal{G} = \mathcal{G}_S$ for a matrix function S ; the missing property is *shift-invariance*, i.e., one must require in addition that \mathcal{G} is invariant under multiplication by the coordinate function $\chi(\lambda) = \frac{\lambda-1}{\lambda+1}$. Then one gets that T and M_{χ} commute and one can conclude that T is a multiplication operator: $T = M_S$ for some multiplier function S . Thus the issue is to construct maximal negative subspaces of $\mathcal{M}_{\mathfrak{D}}$ (which are then also maximal negative as subspaces of \mathcal{K} by Lemma 3.2) which are also shift-invariant.

To achieve this goal, it is convenient to assume that $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$ is strictly positive, i.e., that $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$ is a Hilbert space. It then follows in particular that $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$ is regular, i.e., $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$ and its J -orthogonal complement (relative to \mathcal{K}) $\mathcal{M}_{\mathfrak{D}}$ form a J -orthogonal decomposition of \mathcal{K} :

$$\mathcal{K} = \mathcal{M}_{\mathfrak{D}}[+]_J \mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}.$$

One can argue that one can use an approximation/normal-families argument to reduce the general case to this special case, but we do not go into details on this point here. Then results from [10] imply that there is a J -Beurling-Lax representer for $\mathcal{M}_{\mathfrak{D}}$, i.e., there is a J -phase function

$$\Theta \in L^2_{(p+m) \times (p+m)}(i\mathbb{R}) \quad \text{with} \quad \Theta(\lambda)^* J \Theta(\lambda) = J \quad \text{for a.e. } \lambda \in \Pi_+$$

such that (2.29) holds. As both

$$\mathcal{M}_{\mathfrak{D}} \ominus (\mathcal{M}_{\mathfrak{D}} \cap H^2_{p+m}(\Pi_+)) \quad \text{and} \quad H^2_{p+m}(\Pi_+) \ominus (H^2_{p+m}(\Pi_+) \cap \mathcal{M}_{\mathfrak{D}})$$

are finite-dimensional, in fact one can show that Θ is rational and bounded on $i\mathbb{R}$. Then the multiplication operator $M_{\Theta}: k \mapsto \Theta \cdot k$ is a Kreĭn-space isomorphism from $H^2_{p+m}(\Pi_+)$ (a Kreĭn space with inner product induced by $J = \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix}$) onto $\mathcal{M}_{\mathfrak{D}}$ which also intertwines the multiplication operator M_{χ} on the respective spaces. It follows that shift-invariant $\mathcal{M}_{\mathfrak{D}}$ -maximal-negative subspaces \mathcal{G} are exactly those of the form

$$\mathcal{G} = \Theta \cdot \begin{bmatrix} G \\ I_m \end{bmatrix} \cdot H^2_m(\Pi_+), \quad \text{where } G \in \mathcal{S}^{p \times m}(\Pi_+).$$

By the preceding analysis, any such subspace \mathcal{G} also has the form

$$\mathcal{G} = \begin{bmatrix} S \\ I_m \end{bmatrix} \cdot \psi^{-1} H^2_m(\Pi_+)$$

where $S \in \mathcal{S}^{p \times m}(\Pi_+)$ is a Schur-class solution of the interpolation conditions (1.11), (1.12), (1.13). Moreover one can reverse this analysis to see that any solution S of the BTOA-NP interpolation problem arises in this way from a $G \in \mathcal{S}^{p \times m}(\Pi_+)$. From the subspace equality

$$\begin{bmatrix} S \\ I_m \end{bmatrix} \cdot \psi^{-1} H^2_m(\Pi_+) = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \cdot \begin{bmatrix} G \\ I_m \end{bmatrix} \cdot H^2_m(\Pi_+)$$

one can solve for S in terms of G : in particular we have

$$\begin{bmatrix} S \\ I_m \end{bmatrix} \cdot \psi^{-1} I_m \in \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \cdot \begin{bmatrix} G \\ I_m \end{bmatrix} \cdot H^2_m(\Pi_+),$$

so there must be a function $Q \in H^{\infty}_{m \times m}(\Pi_+)$ so that

$$\begin{bmatrix} S \\ I_m \end{bmatrix} \cdot \psi^{-1} I_m = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \cdot \begin{bmatrix} G \\ I_m \end{bmatrix} \cdot Q.$$

As we saw in Section 2, the latter equality (which is the same as (2.39)) implies the representation formula (1.20) for the set of solutions S . This completes the proof of Theorem 3.3. \square

Remark 3.4. Note that in this Grassmannian/Kreĭn-space approach we have not even mentioned that the J -phase Θ is actually J -inner (i.e., $\Theta(\lambda)$ is J contractive at its points of analyticity in Π_+); this condition and the winding number argument in the proof via the state-space approach in Section 2 have been replaced by the condition that $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$ is a positive subspace and consequences of this assumption coming out of Lemma 3.2.

4. STATE-SPACE VERSUS GRASSMANNIAN/KREĬN-SPACE-GEOMETRY SOLUTION CRITERIA

Assume that we are given a Π_+ -admissible interpolation data set \mathfrak{D} with $\Gamma_{\mathfrak{D}}$ invertible. When we combine the results of Theorems 1.2, 3.3 and 2.3, we see immediately that $\Gamma_{\mathfrak{D}} \succ 0$ if and only if the subspace $\mathcal{M}_{\mathfrak{D}}^{[\perp]}$ is positive as a subspace of the Kreĭn-space \mathcal{K} (3.2), since each of these two conditions is equivalent to the existence of solutions for the BTOA-NP interpolation problem with data set \mathfrak{D} . It is not too much of a stretch to speculate that the strict positive definiteness of $\Gamma_{\mathfrak{D}}$ is equivalent to strict positivity of $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$. Furthermore, in the case where $\Gamma_{\mathfrak{D}}$ is invertible, by the analysis in Section 2.2 we know that positive-definiteness of $\Gamma_{\mathfrak{D}}$ is equivalent to positivity of the kernel $K_{\Theta, J}$ (2.20), or to the reproducing kernel space $\mathcal{H}(K_{\Theta, J})$ being a Hilbert space. The goal of this section is to carry out some additional geometric analysis to verify these equivalences for the nondegenerate case ($\Gamma_{\mathfrak{D}}$ invertible) directly.

Corollary 4.1. *Suppose that \mathfrak{D} is a Π_+ -admissible BTOA interpolation data set, let $\Gamma_{\mathfrak{D}}$ be the matrix given in (1.18) and let $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]} \subset \mathcal{K}$ be the subspace defined in (3.3). Then the following are equivalent:*

- (1) $\Gamma_{\mathfrak{D}} \succ 0$.
- (2) $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$ is a strictly positive subspace of \mathcal{K} (i.e., $\mathcal{M}^{[\perp \mathcal{K}]}$ is a Hilbert space in the J -inner product).
- (3) The reproducing kernel Pontryagin space $\mathcal{H}(K_{\Theta, J})$ is actually a Hilbert space.

Proof. For simplicity we consider first the case where the data set \mathfrak{D} has the form

$$\mathfrak{D}_L = (Z, X, Y; \emptyset, \emptyset, \emptyset; \emptyset), \quad (4.1)$$

i.e., there are only Left Tangential interpolation conditions (1.11).

Case 1: The LTOA setting. In case \mathfrak{D} has the form $\mathfrak{D} = \mathfrak{D}_L$ as in (4.1), the matrix $\Gamma_{\mathfrak{D}}$ collapses down to $\Gamma_{\mathfrak{D}_L} = \Gamma_L$ and $\mathcal{M}_{\mathfrak{D}_L}$ collapses down to

$$\mathcal{M}_{\mathfrak{D}_L} = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in H_{p+m}^2(\Pi_+): \left(\begin{bmatrix} X & -Y \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \right)^{\wedge L} (Z) = 0 \right\}.$$

Furthermore, in the present case, $\mathcal{M}_{\mathfrak{D}_L, -} = H_m^2(\Pi_+)$ and therefore, \mathcal{K} given by (3.2) is simply $\mathcal{K} = \begin{bmatrix} L_p^2(i\mathbb{R}) \\ H_m^2(\Pi_+) \end{bmatrix}$.

We view the map $\begin{bmatrix} f \\ g \end{bmatrix} \mapsto ([X \ -Y] \begin{bmatrix} f \\ g \end{bmatrix})^{\wedge L}(Z)$ as an operator

$$\mathcal{C}_{Z,[X \ -Y]}: H_{p+m}^2(\Pi_+) \rightarrow \mathbb{C}^{n_Z}$$

which can be written out more explicitly as an integral operator along the imaginary line:¹

$$\mathcal{C}_{Z,[X \ -Y]}: \begin{bmatrix} f_+ \\ f_- \end{bmatrix} \mapsto \frac{1}{2\pi} \int_{-\infty}^{\infty} -(iyI - Z)^{-1} [X \ -Y] \begin{bmatrix} f_+(iy) \\ f_-(iy) \end{bmatrix} dy.$$

Then we can view $\mathcal{M}_{\mathfrak{D}_L}$ as an operator kernel:

$$\mathcal{M}_{\mathfrak{D}_L} = \text{Ker } \mathcal{C}_{Z,[X \ -Y]}.$$

We are actually interested in the J -orthogonal complement

$$\begin{aligned} \mathcal{M}_{\mathfrak{D}_L}^{[\perp \mathcal{K}]} &:= \mathcal{K}[-]_J \mathcal{M}_{\mathfrak{D}_L} = \begin{bmatrix} L_p^2(i\mathbb{R}) \\ H_m^2(\Pi_+) \end{bmatrix} [-]_J \mathcal{M}_{\mathfrak{D}_L} \\ &= \begin{bmatrix} H_p^2(\Pi_-) \\ 0 \end{bmatrix} \oplus (H_{p+m}^2(\Pi_+) [-]_J \mathcal{M}_{\mathfrak{D}_L}). \end{aligned}$$

As the subspace $\begin{bmatrix} H_p^2(\Pi_-) \\ 0 \end{bmatrix}$ is clearly positive, we see that $\mathcal{M}_{\mathfrak{D}_L}^{[\perp \mathcal{K}]}$ is positive if and only if its subspace

$$\mathcal{M}_{\mathfrak{D}_L}^{[\perp H_{p+m}^2(\Pi_+)]} := H_{p+m}^2(\Pi_+) [-]_J \mathcal{M}_{\mathfrak{D}_L}$$

is positive. By standard operator-theory duality, we can express the latter (finite-dimensional and hence closed) subspace as an operator range:

$$\mathcal{M}_{\mathfrak{D}_L}^{[\perp H_{p+m}^2(\Pi_+)]} = \text{Ran } J (\mathcal{C}_{Z,[X \ -Y]})^*,$$

where the adjoint is with respect to the standard Hilbert-space inner product on $H_{p+m}^2(\Pi_+)$ and the standard Euclidean inner product on \mathbb{C}^{n_Z} . One can compute the adjoint $(\mathcal{C}_{Z,[X \ -Y]})^*: \mathbb{C}^{n_Z} \rightarrow \begin{bmatrix} H_p^2(\Pi_+) \\ H_m^2(\Pi_+) \end{bmatrix}$ explicitly as

$$(\mathcal{C}_{Z,[X \ -Y]})^*: x \mapsto \begin{bmatrix} -X^* \\ Y^* \end{bmatrix} (\lambda I + Z^*)^{-1} x.$$

Then the Kreĭn-space orthogonal complement $H_{p+m}^2(\Pi_+) [-]_J \mathcal{M}_{\mathfrak{D}_L}$ can be identified with

$$\begin{aligned} \mathcal{M}_{\mathfrak{D}_L}^{[\perp H_{p+m}^2(\Pi_+)]} &= J \cdot \text{Ran}(\mathcal{C}_{Z,[X \ -Y]})^* \\ &= \left\{ \begin{bmatrix} -X^* \\ -Y^* \end{bmatrix} (\lambda I + Z^*)^{-1} x : x \in \mathbb{C}^{n_Z} \right\}. \end{aligned} \quad (4.2)$$

To characterize when $\mathcal{M}_{\mathfrak{D}_L}^{[\perp H_{p+m}^2(\Pi_+)]}$ is a positive subspace, it suffices to compute the Kreĭn-space inner-product gramian matrix \mathbb{G} for $\mathcal{M}_{\mathfrak{D}_L}^{[\perp H_{p+m}^2(\Pi_+)]}$

¹We view operators of this form as *control-like* operators; they and their cousins (*observer-like* operators) will be discussed in a broader context as part of the analysis of Case 2 to come below.

with respect to its parametrization by \mathbb{C}^{n_Z} in (4.2):

$$\begin{aligned} & \langle \mathbb{G}x, x' \rangle_{\mathbb{C}^{n_Z}} \\ &= \frac{1}{2\pi} \left\langle J \begin{bmatrix} -X^* \\ -Y^* \end{bmatrix} (\lambda I + Z^*)^{-1} x, \begin{bmatrix} -X^* \\ -Y^* \end{bmatrix} (\lambda I + Z^*)^{-1} x' \right\rangle_{H_{p+m}^2(\Pi_+)} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle (-iyI + Z)^{-1} (XX^* - YY^*) (iyI + Z^*)^{-1} x, x' \rangle_{\mathbb{C}^{n_Z}} dy. \end{aligned}$$

Thus \mathbb{G} is given by

$$\mathbb{G} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-iyI + Z)^{-1} (XX^* - YY^*) (iyI + Z^*)^{-1} dy.$$

Introduce the change of variable $\zeta = iy$, $d\zeta = i dy$ to write this as a complex line integral

$$\begin{aligned} \mathbb{G} &= \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\Gamma_{R,1}} (-\zeta I + Z)^{-1} (XX^* - YY^*) (\zeta I + Z^*)^{-1} d\zeta \\ &= \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{-\Gamma_{R,1}} (\zeta I - Z)^{-1} (XX^* - YY^*) (\zeta I + Z^*)^{-1} d\zeta \end{aligned}$$

where $\Gamma_{R,1}$ is the straight line from $-iR$ to iR and $-\Gamma_{R,1}$ is the same path but with reverse orientation (the straight line from iR to $-iR$). Since the integrand

$$f(\zeta) = (\zeta I - Z)^{-1} (XX^* - YY^*) (\zeta I + Z^*)^{-1} \quad (4.3)$$

satisfies an estimate of the form $\|f(\zeta)\| \leq \frac{M}{|\zeta|^2}$ as $|\zeta| \rightarrow \infty$, it follows that

$$\lim_{R \rightarrow \infty} \int_{\Gamma_{R,2}} (\zeta I - Z)^{-1} (XX^* - YY^*) (\zeta I + Z^*)^{-1} d\zeta = 0$$

where $\Gamma_{R,2}$ is the semicircle of radius R with counterclockwise orientation starting at the point $-iR$ and ending at the point iR (parametrization: $\zeta = Re^{i\theta}$ with $-\pi/2 \leq \theta \leq \pi/2$). Hence we see that

$$\mathbb{G} = \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\Gamma_R} (\zeta I - Z)^{-1} (XX^* - YY^*) (\zeta I + Z^*)^{-1} d\zeta$$

where Γ_R is the simple closed curve $-\Gamma_{R,1} + \Gamma_{R,2}$. By the residue theorem, this last expression is independent of R once R is so large that all the RHP poles of the integrand $f(\zeta)$ (4.3) are inside the curve Γ_R , and hence

$$\mathbb{G} = \frac{1}{2\pi i} \int_{\Gamma_R} (\zeta I - Z)^{-1} (XX^* - YY^*) (\zeta I + Z^*)^{-1} d\zeta$$

for any R large enough. This enables us to compute \mathbb{G} via residues:

$$\mathbb{G} = \sum_{z_0 \in \Pi_+} \text{Res}_{\zeta=z_0} (\zeta I - Z)^{-1} (XX^* - YY^*) (\zeta I + Z^*)^{-1}. \quad (4.4)$$

We wish to verify that \mathbb{G} satisfies the Lyapunov equation

$$\mathbb{G}Z^* + Z\mathbb{G} = XX^* - YY^*. \quad (4.5)$$

Toward this end let us first note that

$$\begin{aligned} & (\zeta I - Z)^{-1} A (\zeta I + Z^*)^{-1} Z^* + Z (\zeta I - Z)^{-1} A (\zeta I + Z^*)^{-1} \\ &= (\zeta I - Z)^{-1} A - A (\zeta I + Z^*)^{-1} \end{aligned}$$

for any $A \in \mathbb{C}^{n_Z \times n_Z}$. Making use of the latter equality with $A = XX^* - YY^*$ we now deduce from the formula (4.4) for \mathbb{G} that

$$\begin{aligned} \mathbb{G}Z^* + Z\mathbb{G} &= \sum_{z_0 \in \Pi_+} \text{Res}_{\zeta=z_0} ((\zeta I - Z)^{-1} (XX^* - YY^*) \\ &\quad - (XX^* - YY^*)(\zeta I + Z^*)^{-1}) \\ &= I \cdot (XX^* - YY^*) - (XX^* - YY^*) \cdot 0 = XX^* - YY^* \end{aligned}$$

where for the last step we use that Z has all its spectrum in the right half plane while $-Z^*$ has all its spectrum in the left half plane; also note that in general the sum of the residues of any resolvent matrix $R(\zeta) = (\zeta I - A)^{-1}$ is the identity matrix, due to the Laurent expansion at infinity for $R(\zeta)$: $R(\zeta) = \sum_{n=0}^{\infty} A^n \zeta^{-n-1}$. This completes the verification of (4.5).

Since both Γ_L and \mathbb{G} satisfy the same Lyapunov equation (1.16) which has a unique solution since $\sigma(Z) \cap \sigma(-Z^*) = \emptyset$, we conclude that $\mathbb{G} = \Gamma_L$. This completes the direct proof of the equivalence of conditions (1) and (2) in Corollary 4.1 for the case that $\mathfrak{D} = \mathfrak{D}_L$.

To make the connection with the kernel $K_{\Theta, J}$, we note that there is a standard way to identify a reproducing kernel Hilbert space $\mathcal{H}(K)$ of a particular form with an operator range (see e.g. [39] or [7]). Specifically, let M_{Θ} be the multiplication operator

$$M_{\Theta}: f(\lambda) \mapsto \Theta(\lambda)f(\lambda)$$

acting on $H_{p+m}^2(\Pi_+)$, identify J with $J \otimes I_{H^2(\Pi_+)}$ acting on $H_{p+m}^2(\Pi_+)$, and define $W \in \mathcal{L}(H_{p+m}^2(\Pi_+))$ by

$$W = J - M_{\Theta}J(M_{\Theta})^*.$$

For $w \in \Pi_+$ and $\mathbf{y} \in \mathbb{C}^{p+m}$, let $k_{w, \mathbf{y}}(z) = \frac{1}{z - \overline{w}} \mathbf{y}$ by the kernel element associate with the Szegő kernel $k_{\text{Sz}} \otimes I_{\mathbb{C}^{p+m}}$. One can verify

$$Wk_{w, \mathbf{y}} = K_{\Theta, J}(\cdot, w)\mathbf{y} \in \mathcal{H}(K_{\Theta, J}),$$

and furthermore,

$$\begin{aligned} \langle Wk_{w_j, \mathbf{y}_j}, Wk_{w_i, \mathbf{y}_i} \rangle_{\mathcal{H}(K_{\Theta, J})} &= \langle K_{\Theta, J}((w_i, w_j)\mathbf{y}_j, \mathbf{y}_i) \rangle_{\mathbb{C}^{p+m}} \\ &= \langle Wk_{w_j, \mathbf{y}_j}, k_{w_i, \mathbf{y}_i} \rangle_{H_{p+m}^2(\Pi_+)}. \end{aligned}$$

As Θ is rational and M_{Θ} is a J -isometry, one can see that $\text{Ran } W$ is already closed. Hence we have the concrete identification $\mathcal{H}(K_{\Theta, J}) = \text{Ran } W$ with lifted inner product

$$\langle Wf, Wg \rangle_{\mathcal{H}(K_{\Theta, J})} = \langle Wf, g \rangle_{H_{p+m}^2(\Pi_+)}.$$

As M_Θ is a J -isometry, the operator $M_\Theta J(M_\Theta)^* =: M_\Theta(M_\Theta)^{[*]}$ is the J -selfadjoint projection onto $\Theta \cdot H_{p+m}^2(\Pi_+)$ and $WJ = I - M_\Theta(M_\Theta)^{[*]}$ is the J -self-adjoint projection onto $H_{p+m}^2[-] \Theta \cdot H_{p+m}^2(\Pi_+) = \mathcal{M}_\Theta^{[\perp \mathcal{K}]}$. We then see that, for all $f, g \in H_{p+m}^2(\Pi_+)$,

$$\langle WJf, WJg \rangle_{\mathcal{H}(K_{\Theta,J})} = \langle WJf, Jg \rangle_{H_{p+m}^2(\Pi_+)} = \langle J \cdot WJf, WJg \rangle_{H_{p+m}^2(\Pi_+)},$$

i.e., the identity map is a Kreĭn-space isomorphism between $\mathcal{H}(K_{\Theta,J})$ and $\mathcal{M}_\Theta^{[\perp \mathcal{K}]}$ with the J -inner product. In particular, we arrive at the equivalence of conditions (2) and (3) in Corollary 4.1 for Case 1.

Case 2: The general BTOA setting: To streamline formulas to come, we introduce two types of control-like operators and two types of observer-like operators as follows (for fuller details and systems-theory motivation, we refer to [13] for the discrete-time setting and [3] for the continuous-time setting). Suppose that (A, B) is an input pair of matrices (so A has, say, size $N \times N$ and B has size $N \times n$). We assume that either A is stable ($\sigma(A) \subset \Pi_-$) or A is antistable ($\sigma(A) \subset \Pi_+$). In case $\sigma(A) \subset \Pi_+$, we define a control-like operator as appeared in the Case 1 analysis

$$\mathcal{C}_{A,B}: H_n^2(\Pi_+) \rightarrow \mathbb{C}^N$$

by

$$\begin{aligned} \mathcal{C}_{A,B}: g \mapsto (Bg)^{\wedge L}(A) &:= \sum_{z \in \Pi_+} \text{Res}_{\lambda=z} (\lambda I - A)^{-1} Bg(\lambda) \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} (iyI - A)^{-1} Bg(iy) dy. \end{aligned}$$

In case $\sigma(A) \subset \Pi_-$, we define a complementary control-like operator

$$\mathcal{C}_{A,B}^\times: H_n^2(\Pi_-) \rightarrow \mathbb{C}^N$$

by

$$\begin{aligned} \mathcal{C}_{A,B}^\times: g \mapsto (Bg)^{\wedge L}(A) &:= \sum_{z \in \Pi_-} \text{Res}_{\lambda=z} (\lambda I - A)^{-1} Bg(\lambda) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (iyI - A)^{-1} Bg(iy) dy. \end{aligned}$$

Suppose next that (C, A) is an output-pair, say of respective sizes $n \times N$ and $N \times N$, and that A is either stable or antistable. In case A is antistable ($\sigma(A) \subset \Pi_+$), we define the observer-like operator

$$\mathcal{O}_{C,A}: \mathbb{C}^N \rightarrow H_n^2(\Pi_-)$$

by

$$\mathcal{O}_{C,A}: x \mapsto C(\lambda I - A)^{-1}x.$$

In case A is stable (so $\sigma(A) \subset \Pi_-$), then the complementary observer-like operator is given by the same formula but maps to the complementary H^2

space:

$$\mathcal{O}_{C,A}^\times: \mathbb{C}^N \mapsto H_n^2(\Pi_+)$$

given again by

$$\mathcal{O}_{C,A}^\times: x \mapsto C(\lambda I - A)^{-1}x.$$

We are primarily interested in the case where A is antistable and we consider the operators $\mathcal{C}_{A,B}: H_n^2(\Pi_+) \rightarrow \mathbb{C}^N$ and $\mathcal{O}_{C,A}: \mathbb{C}^N \mapsto H_n^2(\Pi_-)$. However a straightforward exercise is to show that the complementary operators come up when computing adjoints: for A antistable, $-A^*$ is stable and we have the formulas

$$(\mathcal{O}_{C,A})^* = -\mathcal{C}_{-A^*,C^*}^\times: H_n^2(\Pi_-) \mapsto \mathbb{C}^N, \quad (\mathcal{C}_{A,B})^* = \mathcal{O}_{B^*,-A^*}^\times: \mathbb{C}^N \mapsto H_n^2(\Pi_+).$$

Assume now that $\mathcal{M}_{\mathfrak{D}} \subset L_{p+m}^2(\Pi_+)$ is defined as in (2.25) for a Π_+ -admissible interpolation data set $\mathfrak{D} = (U, V, W; Z, X, Y; \Gamma)$. Thus (U, W) and (V, W) are output pairs with $\sigma(W) \subset \Pi_+$ and (Z, X) and (Z, Y) are input pairs with $\sigma(Z) \subset \Pi_+$. We therefore have observer-like and control-like operators

$$\begin{aligned} \mathcal{O}_{V,W}: \mathbb{C}^{n_W} &\rightarrow H_p^2(\Pi_-), & \mathcal{O}_{U,W}: \mathbb{C}^{n_W} &\rightarrow H_m^2(\Pi_-), \\ \mathcal{C}_{Z,X}: H_p^2(\Pi_+) &\rightarrow \mathbb{C}^{n_Z}, & \mathcal{C}_{Z,Y}: H_m^2(\Pi_+) &\rightarrow \mathbb{C}^{n_Z} \end{aligned}$$

defined as above, as well as the observer-like and control-like operators

$$\mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W} := \begin{bmatrix} \mathcal{O}_{V,W} \\ \mathcal{O}_{U,W} \end{bmatrix}, \quad \mathcal{C}_{Z, \begin{bmatrix} X & -Y \end{bmatrix}} = \begin{bmatrix} \mathcal{C}_{Z,X} & -\mathcal{C}_{Z,Y} \end{bmatrix}.$$

Then the adjoint operators have the form

$$\begin{aligned} (\mathcal{O}_{V,W})^* &= -\mathcal{C}_{-W^*,V^*}^\times: H_p^2(\Pi_-) \rightarrow \mathbb{C}^{n_W}, \\ (\mathcal{O}_{U,W})^* &= -\mathcal{C}_{-W^*,U^*}^\times: H_m^2(\Pi_-) \rightarrow \mathbb{C}^{n_W}, \\ (\mathcal{C}_{Z,X})^* &= \mathcal{O}_{X^*,-Z^*}^\times: \mathbb{C}^{n_Z} \rightarrow H_p^2(\Pi_+), \\ (\mathcal{C}_{Z,Y})^* &= \mathcal{O}_{Y^*,-Z^*}^\times: \mathbb{C}^{n_Z} \rightarrow H_m^2(\Pi_+) \end{aligned}$$

and are given explicitly by:

$$\begin{aligned} (\mathcal{O}_{V,W})^*: g_1 &\mapsto -\frac{1}{2\pi} \int_{-\infty}^{\infty} (iyI + W^*)^{-1} V^* g_1(iy) dy, \\ (\mathcal{O}_{U,W})^*: g_2 &\mapsto -\frac{1}{2\pi} \int_{-\infty}^{\infty} (iyI + W^*)^{-1} U^* g_2(iy) dy, \\ (\mathcal{C}_{Z,X})^*: x &\mapsto X^*(\lambda I + Z^*)^{-1}x, \quad (\mathcal{C}_{Z,Y})^*: x \mapsto Y^*(\lambda I + Z^*)^{-1}x. \end{aligned}$$

Furthermore one can check via computations as in the derivation of (4.4) above that the J -observability and J -controllability gramians

$$\begin{aligned} \mathcal{G}_{Z, \begin{bmatrix} X & -Y \end{bmatrix}}^J &:= \mathcal{C}_{Z,X} \mathcal{C}_{Z,X}^* - \mathcal{C}_{Y,Z} \mathcal{C}_{Z,Y}^* =: \mathcal{G}_{Z,X} - \mathcal{G}_{Z,Y}, \\ \mathcal{G}_{\begin{bmatrix} V \\ U \end{bmatrix}, W}^J &:= \mathcal{O}_{V,W}^* \mathcal{O}_{V,W} - \mathcal{O}_{U,W}^* \mathcal{O}_{U,W} =: \mathcal{G}_{V,W} - \mathcal{G}_{U,W} \end{aligned}$$

satisfy the respective Lyapunov equations

$$\begin{aligned}\mathcal{G}_{Z,[X-Y]}^J Z^* + Z \mathcal{G}_{Z,[X-Y]}^J &= X X^* - Y Y^*, \\ \mathcal{G}_{\begin{bmatrix} V \\ U \end{bmatrix}, W}^J W + W^* \mathcal{G}_{\begin{bmatrix} V \\ U \end{bmatrix}, W}^J &= V^* V - U^* U.\end{aligned}$$

Hence, by the uniqueness of such solutions and the characterizations of Γ_L and Γ_R in (1.16), (1.17), we get

$$\mathcal{G}_{[X-Y], Z}^J = \Gamma_L, \quad \mathcal{G}_{\begin{bmatrix} V \\ U \end{bmatrix}, W}^J = -\Gamma_R. \quad (4.6)$$

Then the representation (2.25) for $\mathcal{M}_{\mathfrak{D}}$ can be rewritten more succinctly as

$$\begin{aligned}\mathcal{M}_{\mathfrak{D}} = \left\{ \mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W} x + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : x \in \mathbb{C}^{n_W} \text{ and } \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in H_{p+m}^2(\Pi_+) \right. \\ \left. \text{such that } \mathcal{C}_{Z,[X-Y]} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \Gamma x \right\}. \quad (4.7)\end{aligned}$$

It is readily seen from the latter formula that

$$\begin{aligned}P_{H_{p+m}^2(\Pi_-)} \mathcal{M}_{\mathfrak{D}} &= \text{Ran } \mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W}, \\ \mathcal{M}_{\mathfrak{D}} \cap H_{p+m}^2(\Pi_+) &= \text{Ker } \mathcal{C}_{Z,[X-Y]},\end{aligned} \quad (4.8)$$

and therefore,

$$\mathcal{M}_{\mathfrak{D}} \cap \begin{bmatrix} H_p^2(\Pi_+) \\ 0 \end{bmatrix} = \begin{bmatrix} \text{Ker } \mathcal{C}_{Z,X} \\ 0 \end{bmatrix}, \quad \mathcal{M}_{\mathfrak{D}} \cap \begin{bmatrix} 0 \\ H_m^2(\Pi_+) \end{bmatrix} = \begin{bmatrix} 0 \\ \text{Ker } \mathcal{C}_{Z,Y} \end{bmatrix}.$$

Lemma 4.2. *If $\mathcal{M}_{\mathfrak{D}}$ is given by (4.7), then the J -orthogonal complement $\mathcal{M}_{\mathfrak{D}}^{[\perp]} = L_{p+m}^2(i\mathbb{R})[-]_J \mathcal{M}_{\mathfrak{D}}$ with respect to the space $L_{p+m}^2(i\mathbb{R})$ is given by*

$$\begin{aligned}\mathcal{M}_{\mathfrak{D}}^{[\perp]} = \left\{ J(\mathcal{C}_{Z,[X-Y]})^* y + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} : y \in \mathbb{C}^{n_Z} \text{ and } \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in H_{p+m}^2(\Pi_-) \right. \\ \left. \text{such that } (\mathcal{O}_{V,W})^* g_1 - (\mathcal{O}_{U,W})^* g_2 = -\Gamma^* y \right\}. \quad (4.9)\end{aligned}$$

Proof. Since $\mathcal{M}_{\mathfrak{D}}^{[\perp]}$ is J -orthogonal to $\mathcal{M}_{\mathfrak{D}} \cap H_{p+m}^2(\Pi_+) = \text{Ker } \mathcal{C}_{Z,[X-Y]}$, it follows that $P_{H_{p+m}^2(\Pi_+)} \mathcal{M}_{\mathfrak{D}}^{[\perp]}$ is also J -orthogonal to $\text{Ker } \mathcal{C}_{Z,[X-Y]}$. Hence $P_{H_{p+m}^2(\Pi_+)} \mathcal{M}_{\mathfrak{D}}^{[\perp]} \subset J \text{Ran}((\mathcal{C}_{Z,[X-Y]})^*)$ and each $\mathbf{g} \in \mathcal{M}_{\mathfrak{D}}^{[\perp]}$ has the form

$$\mathbf{g} = J(\mathcal{C}_{Z,[X-Y]})^* y + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \quad \text{with } y \in \mathbb{C}^{n_Z} \text{ and } \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in H_{p+m}^{2\perp}(\Pi_+).$$

For such an element to be in $\mathcal{M}_{\mathfrak{D}}^{[\perp]}$, we compute the J -inner product of such an element against a generic element of $\mathcal{M}_{\mathfrak{D}}$: for all $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in H_{p+m}^2(\Pi_+)$ and $x \in \mathbb{C}^{n_Z}$ such that $\mathcal{C}_{Z,X} f_1 - \mathcal{C}_{Z,Y} f_2 = \Gamma x$, we must have

$$\begin{aligned}0 &= \left\langle J \left(J(\mathcal{C}_{Z,[X-Y]})^* y + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right), \mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W} x + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\rangle_{L_{p+m}^2(i\mathbb{R})} \\ &= \langle y, \mathcal{C}_{Z,X} f_1 - \mathcal{C}_{Z,Y} f_2 \rangle_{\mathbb{C}^{n_Z}} + \langle (\mathcal{O}_{V,W})^* g_1 - (\mathcal{O}_{U,W})^* g_2, x \rangle_{\mathbb{C}^{n_W}} \\ &= \langle y, \Gamma x \rangle_{\mathbb{C}^{n_Z}} + \langle (\mathcal{O}_{V,W})^* g_1 - (\mathcal{O}_{U,W})^* g_2, x \rangle_{\mathbb{C}^{n_W}}\end{aligned}$$

which leads to the coupling condition $(\mathcal{O}_{V,W})^*g_1 - (\mathcal{O}_{U,W})^*g_2 = -\Gamma^*y$ in (4.9). \square

As a consequence of the representation (4.9) we see that

$$\begin{aligned} P_{H_{p+m}^2(\Pi_+)} \mathcal{M}_{\mathfrak{D}}^{[\perp]} &= \text{Ran } J(\mathcal{C}_{Z,[X-Y]})^*, \\ \mathcal{M}_{\mathfrak{D}}^{[\perp]} \cap H_{p+m}^2(\Pi_-) &= \text{Ker } [(\mathcal{O}_{V,W})^* \quad -(\mathcal{O}_{U,W})^*] \end{aligned} \quad (4.10)$$

and therefore,

$$\mathcal{M}_{\mathfrak{D}}^{[\perp]} \cap \left[H_{p+m}^2(\Pi_-) \right] = \left[\text{Ker}(\mathcal{O}_{V,W})^* \right], \quad \mathcal{M}_{\mathfrak{D}}^{[\perp]} \cap \left[H_m^0(\Pi_-) \right] = \left[\text{Ker}(\mathcal{O}_{U,W})^* \right].$$

In this section we shall impose an additional assumption:

Nondegeneracy assumption: *Not only $\mathcal{M}_{\mathfrak{D}}$ but also $\mathcal{M}_{\mathfrak{D}} \cap H_{p+m}^2(\Pi_+)$ and $\mathcal{M}_{\mathfrak{D}}^{[\perp]} \cap H_{p+m}^2(\Pi_-)$ (see (4.8) and (4.10)) are regular subspaces (i.e., have good Kreĭn-space orthogonal complements—as explained in Section 3.1) of the Kreĭn space $L_{p+m}^2(\Pi_+)$ (with the J -inner product).*

We proceed via a string of lemmas.

Lemma 4.3. (1) *The space $\mathcal{M}_{\mathfrak{D}}$ given in (4.7) decomposes as*

$$\mathcal{M}_{\mathfrak{D}} = \widehat{\mathbb{G}}_T[+] \mathcal{M}_{\mathfrak{D},1}[+] \mathcal{M}_{\mathfrak{D},2}, \quad (4.11)$$

where

$$\begin{aligned} \widehat{\mathbb{G}}_T &= \mathcal{M}_{\mathfrak{D}}[-]_J \text{Ker } \mathcal{C}_{Z,[X-Y]}, \\ \mathcal{M}_{\mathfrak{D},1} &= \text{Ker } \mathcal{C}_{Z,[X-Y]}[-]_J \left(\left[\text{Ker } \mathcal{C}_{Z,X} \right] \oplus \left[\text{Ker } \mathcal{C}_{Z,Y} \right] \right), \\ \mathcal{M}_{\mathfrak{D},2} &= \left[\text{Ker } \mathcal{C}_{Z,X} \right] \oplus \left[\text{Ker } \mathcal{C}_{Z,Y} \right]. \end{aligned} \quad (4.12)$$

More explicitly, the operator $T : \text{Ran } \mathcal{O}_{[V],W} \rightarrow \text{Ran } J(\mathcal{C}_{Z,[X-Y]})^*$ is uniquely determined by the identity

$$\mathcal{C}_{Z,[X-Y]} T \mathcal{O}_{[V],W} = -\Gamma, \quad (4.13)$$

and $\widehat{\mathbb{G}}_T$ is the graph space for $-T$ parametrized as

$$\begin{aligned} \widehat{\mathbb{G}}_T &= \left\{ -\mathbf{f} + T\mathbf{f} : \mathbf{f} \in \text{Ran } \mathcal{O}_{[V],W} \right\} \\ &= \left\{ -\mathcal{O}_{[V],W}x + T\mathcal{O}_{[V],W}x : x \in \mathbb{C}^{n_W} \right\}, \end{aligned} \quad (4.14)$$

while $\mathcal{M}_{\mathfrak{D},1}$ is given explicitly by

$$\mathcal{M}_{\mathfrak{D},1} = \text{Ran } \begin{bmatrix} (\mathcal{C}_{Z,X})^* (\mathcal{G}_{Z,X})^{-1} \mathcal{G}_{Z,Y} \\ (\mathcal{C}_{Z,Y})^* \end{bmatrix}. \quad (4.15)$$

(2) *Dually, the subspace $\mathcal{M}_{\mathfrak{D}}^{[\perp]} = L_{p+m}^2(i\mathbb{R})[-]_J \mathcal{M}_{\mathfrak{D}}$ decomposes as*

$$\mathcal{M}_{\mathfrak{D}}^{[\perp]} = \mathbb{G}_{T[*]}[+](\mathcal{M}_{\mathfrak{D}}^{[\perp]})_1[+](\mathcal{M}_{\mathfrak{D}}^{[\perp]})_2, \quad (4.16)$$

where

$$\begin{aligned}\mathbb{G}_{T[*]} &= \mathcal{M}_{\mathfrak{D}}^{[\perp]}[-]_J \operatorname{Ker} [(\mathcal{O}_{V,W})^* \quad -(\mathcal{O}_{U,W})^*], \\ (\mathcal{M}_{\mathfrak{D}}^{[\perp]})_1 &= \operatorname{Ker} [(\mathcal{O}_{V,W})^* \quad -(\mathcal{O}_{U,W})^*] [-]_J \left(\begin{bmatrix} \operatorname{Ker}(\mathcal{O}_{V,W})^* \\ 0 \end{bmatrix} \oplus \begin{bmatrix} \operatorname{Ker}(\mathcal{O}_{U,W})^* \\ 0 \end{bmatrix} \right), \\ (\mathcal{M}_{\mathfrak{D}}^{[\perp]})_2 &= \begin{bmatrix} \operatorname{Ker}(\mathcal{O}_{V,W})^* \\ 0 \end{bmatrix} \oplus \begin{bmatrix} \operatorname{Ker}(\mathcal{O}_{U,W})^* \\ 0 \end{bmatrix}.\end{aligned}\quad (4.17)$$

More explicitly,

$$\begin{aligned}\mathbb{G}_{T[*]} &= \left\{ \mathbf{g} + T^{[*]} \mathbf{g} : \mathbf{g} \in \operatorname{Ran} J(\mathcal{C}_{Z,[X-Y]})^* \right\} \\ &= \left\{ J(\mathcal{C}_{Z,[X-Y]})^* x + T^{[*]} J(\mathcal{C}_{Z,[X-Y]})^* x : x \in \mathbb{C}^{nz} \right\}\end{aligned}$$

where $T^{[*]} = JT^*J : \operatorname{Ran} J(\mathcal{C}_{Z,[X-Y]})^* \rightarrow \operatorname{Ran} \mathcal{O}_{[V],W}$ is the J -adjoint of T , and

$$(\mathcal{M}_{\mathfrak{D}}^{[\perp]})_1 = \operatorname{Ran} \begin{bmatrix} \mathcal{O}_{V,W} \\ \mathcal{O}_{U,W}(\mathcal{G}_{U,W})^{-1}\mathcal{G}_{V,W} \end{bmatrix}. \quad (4.18)$$

Proof. By the Nondegeneracy Assumption we can define subspaces (4.12) and (4.17), so that $\mathcal{M}_{\mathfrak{D}}$ and $\mathcal{M}_{\mathfrak{D}}^{[\perp]}$ decompose as in (4.11) and (4.16), respectively.

Given an element $\mathbf{g} \in P_{H_{p+m}^2(\Pi_-)} \mathcal{M}_{\mathfrak{D}}$, there is an $\mathbf{f} \in H_{p+m}^2(\Pi_+)$ so that $-\mathbf{g} + \mathbf{f} \in \mathcal{M}_{\mathfrak{D}}$; furthermore, one can choose

$$\mathbf{f} \in H_{p+m}^2(\Pi_+)[-]_J (\mathcal{M}_{\mathfrak{D}} \cap H_{p+m}^2(\Pi_+)) = P_{H_{p+m}^2(\Pi_+)} \mathcal{M}_{\mathfrak{D}}^{[\perp]}.$$

If \mathbf{f}' is another such choice, then $(-\mathbf{g} + \mathbf{f}) - (-\mathbf{g} + \mathbf{f}') = \mathbf{f} - \mathbf{f}'$ is in $\mathcal{M}_{\mathfrak{D}} \cap H_{p+m}^2(\Pi_+)$ as well as in $H_{p+m}^2(\Pi_+)[-]_J (\mathcal{M}_{\mathfrak{D}} \cap H_{p+m}^2(\Pi_+))$. By the Nondegeneracy Assumption, we conclude that $\mathbf{f} = \mathbf{f}'$. Hence there is a well-defined map $\mathbf{g} \mapsto \mathbf{f}$ defining a linear operator T from

$$P_{H_{p+m}^2(\Pi_-)} \mathcal{M}_{\mathfrak{D}} = \operatorname{Ran} \mathcal{O}_{[V],W} \quad \text{into} \quad P_{H_{p+m}^2(\Pi_+)} \mathcal{M}_{\mathfrak{D}}^{[\perp]} = \operatorname{Ran} J(\mathcal{C}_{Z,[X-Y]})^*$$

(see (4.8) and (4.10)). In this way we arrive at a well-defined operator T so that \mathbb{G}_T as in (4.14) is equal to the subspace (see (4.12))

$$\mathcal{M}_{\mathfrak{D}}[-]_J (\mathcal{M}_{\mathfrak{D}} \cap H_{p+m}^2(\Pi_+)) = \mathcal{M}_{\mathfrak{D}}[-]_J \operatorname{Ker} \mathcal{C}_{Z,[X-Y]}.$$

To check that T is also given by (4.13), combine the fact that

$$-\mathcal{O}_{[V],W} x + T \mathcal{O}_{[V],W} x \in \mathcal{M}_{\mathfrak{D}}$$

together with the characterization (4.7) for $\mathcal{M}_{\mathfrak{D}}$ to deduce that

$$\mathcal{C}_{Z,[X-Y]} \cdot T \mathcal{O}_{[V],W} x = -\Gamma x$$

for all x to arrive at (4.13).

To get the formula (4.15), we first note that

$$H_{p+m}^2(\Pi_+)[-]_J \begin{bmatrix} \operatorname{Ker} \mathcal{C}_{Z,X} \\ \operatorname{Ker} \mathcal{C}_{Z,Y} \end{bmatrix} = \begin{bmatrix} \operatorname{Ran}(\mathcal{C}_{Z,X})^* \\ \operatorname{Ran}(\mathcal{C}_{Z,Y})^* \end{bmatrix}. \quad (4.19)$$

The space $\mathcal{M}_{\mathfrak{D},1}$ is the intersection of this space with $\mathcal{M}_{\mathfrak{D}} \cap H_{p+m}^2(\Pi_+)$. Therefore, it consists of elements of the form $\begin{bmatrix} (\mathcal{C}_{Z,X})^* y_1 \\ (\mathcal{C}_{Z,Y})^* y_2 \end{bmatrix}$ subject to condition

$$0 = [\mathcal{C}_{Z,X} \quad -\mathcal{C}_{Z,Y}] \begin{bmatrix} (\mathcal{C}_{Z,X})^* y_1 \\ (\mathcal{C}_{Z,X})^* y_2 \end{bmatrix} = \mathcal{C}_{Z,X}(\mathcal{C}_{Z,X})^* y_1 - \mathcal{C}_{Z,Y}(\mathcal{C}_{Z,Y})^* y_2.$$

By the Π_+ -admissibility requirement on the data set \mathfrak{D} , the gramian $\mathcal{G}_{Z,X} := \mathcal{C}_{Z,X}(\mathcal{C}_{Z,X})^*$ is invertible and hence we may solve this last equation for y_1 :

$$y_1 = \mathcal{G}_{Z,X}^{-1} \mathcal{C}_{Z,Y}(\mathcal{C}_{Z,Y})^* y_2.$$

With this substitution, the element $\begin{bmatrix} (\mathcal{C}_{Z,X})^* y_1 \\ (\mathcal{C}_{Z,Y})^* y_2 \end{bmatrix}$ of the J -orthogonal complement space (4.19) assumes the form

$$\begin{bmatrix} (\mathcal{C}_{Z,X})^* \mathcal{G}_{Z,X}^{-1} \mathcal{C}_{Z,Y}(\mathcal{C}_{Z,Y})^* y_2 \\ (\mathcal{C}_{Z,Y})^* y_2 \end{bmatrix}$$

and we have arrived at the formula (4.15) for $\mathcal{M}_{\mathfrak{D},1}$.

For the dual case (2), similar arguments starting with the representation (4.9) for $\mathcal{M}_{\mathfrak{D}}^{[\perp]}$ show that there is an operator T^\times from $\text{Ran } J(\mathcal{C}_{Z,[X-Y]})^*$ into $\mathcal{M}_{\mathfrak{D}}^{[\perp]}[-]_J \left(\mathcal{M}_{\mathfrak{D}}^{[\perp]} \cap H_{p+m}^2(\Pi_-) \right)$ so that

$$\mathcal{M}_{\mathfrak{D}}^{[\perp]}[-]_J \left(\mathcal{M}_{\mathfrak{D}}^{[\perp]} \cap H_{p+m}^2(\Pi_-) \right) = (I + T^\times) \text{Ran } J(\mathcal{C}_{Z,[X-Y]})^*.$$

From the characterization (4.9) of the space $\mathcal{M}_{\mathfrak{D}}^{[\perp]}$ we see that the condition

$$J(\mathcal{C}_{Z,[X-Y]})^* y + T^\times J(\mathcal{C}_{Z,[X-Y]})^* y \in \mathcal{M}_{\mathfrak{D}}^{[\perp]}$$

requires that, for all $y \in \mathbb{C}^{nz}$,

$$[(\mathcal{O}_{V,W})^* \quad -(\mathcal{O}_{U,W})^*] T^\times \begin{bmatrix} (\mathcal{C}_{Z,X})^* \\ (\mathcal{C}_{Z,Y})^* \end{bmatrix} y = -\Gamma^* y.$$

Cancelling off the vector y and rewriting as an operator equation then gives:

$$\begin{aligned} & [(\mathcal{O}_{V,W})^* \quad -(\mathcal{O}_{U,W})^*] T^\times \begin{bmatrix} (\mathcal{C}_{Z,X})^* \\ (\mathcal{C}_{Z,Y})^* \end{bmatrix} \\ &= [(\mathcal{O}_{V,W})^* \quad (\mathcal{O}_{U,W})^*] J T^\times J \begin{bmatrix} (\mathcal{C}_{Z,X})^* \\ (-\mathcal{C}_{Z,Y})^* \end{bmatrix} \\ &= (\mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W})^* J T^\times J (\mathcal{C}_{Z,[X-Y]})^* = -\Gamma^*. \end{aligned}$$

Taking adjoints of both sides of the identity (4.13) satisfied by T , we see that

$$(\mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W})^* T^* (\mathcal{C}_{Z,[X-Y]})^* = -\Gamma^*.$$

Since $(\mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W})^*$ is injective on the range space of T^\times or $J T^* J$ and $(\mathcal{C}_{Z,[X-Y]})^*$ maps onto the domain space of T^\times or T^* , it follows that $T^\times = J T^* J = T^{[*]}$. The remaining points in statement (2) of the Lemma follow in much the same way as the corresponding points in statement (1). \square

Lemma 4.4. (1) *With \mathcal{K} as in (3.2), the subspace (3.3) decomposes as*

$$\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]} = \mathbb{G}_{T[*]}[+](\mathcal{M}_{\mathfrak{D}}^{[\perp]})_1[+] \left[\begin{smallmatrix} \text{Ker}(\mathcal{O}_{V,W})^* \\ 0 \end{smallmatrix} \right]. \quad (4.20)$$

In particular, $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$ is J -positive if and only if its subspace

$$(\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]})_0 := \mathbb{G}_{T[*]}[+](\mathcal{M}_{\mathfrak{D}}^{[\perp]})_1$$

is J -positive.

(2) *Dually, define a space $\mathcal{K}' \subset L_{p+m}^2(i\mathbb{R})$ by*

$$\mathcal{K}' = \left[\begin{smallmatrix} H_p^2(\Pi_-) \oplus \text{Ran}(\mathcal{C}_{Z,X})^* \\ L_m^2(i\mathbb{R}) \end{smallmatrix} \right]. \quad (4.21)$$

Then $\mathcal{M}_{\mathfrak{D}}^{[\perp]} \subset \mathcal{K}'$ and the space

$$(\mathcal{M}_{\mathfrak{D}}^{[\perp]})^{[\perp \mathcal{K}']} := \mathcal{K}'[-]_J \mathcal{M}_{\mathfrak{D}}^{[\perp]} = \mathcal{K}' \cap \mathcal{M}_{\mathfrak{D}}$$

is given by

$$(\mathcal{M}_{\mathfrak{D}}^{[\perp]})^{[\perp \mathcal{K}']} = \widehat{\mathbb{G}}_T[+]\mathcal{M}_{\mathfrak{D},1}[+] \left[\begin{smallmatrix} 0 \\ \text{Ker} \mathcal{C}_{Z,Y} \end{smallmatrix} \right].$$

In particular, $(\mathcal{M}_{\mathfrak{D}}^{[\perp]})^{[\perp \mathcal{K}]}$ is J -negative if and only if its subspace

$$((\mathcal{M}_{\mathfrak{D}}^{[\perp]})^{[\perp \mathcal{K}']})_0 := \mathbb{G}_T[+]\mathcal{M}_{\mathfrak{D},1}$$

is J -negative.

Proof. By definition, $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]} = \mathcal{K} \cap \mathcal{M}_{\mathfrak{D}}^{[\perp]}$, where $\mathcal{M}_{\mathfrak{D}}^{[\perp]}$ is given by (4.16) and where, due to (3.2) and (2.25), $\mathcal{K} = \left[\begin{smallmatrix} L_p^2(i\mathbb{R}) \\ \text{Ran } \mathcal{O}_{U,W} \oplus H_m^2(\Pi_+) \end{smallmatrix} \right]$. Note that

$$\mathcal{G}_{T[*]} \subset \mathcal{K}, \quad (\mathcal{M}_{\mathfrak{D}}^{[\perp]})_1 \subset H_{p+m}^2(\Pi_+) \subset \mathcal{K},$$

while

$$(\mathcal{M}_{\mathfrak{D}}^{[\perp]})_2 \cap \mathcal{K} = \left[\begin{smallmatrix} \text{Ker}(\mathcal{O}_{V,W})^* \\ \text{Ker}(\mathcal{O}_{U,W})^* \end{smallmatrix} \right] \cap \left[\begin{smallmatrix} L_p^2(i\mathbb{R}) \\ \text{Ran } \mathcal{O}_{U,W} \oplus H_m^2(\Pi_+) \end{smallmatrix} \right] = \left[\begin{smallmatrix} \text{Ker}(\mathcal{O}_{V,W})^* \\ 0 \end{smallmatrix} \right].$$

Putting the pieces together leads to the decomposition (4.20). Since the J -orthogonal summand $\left[\begin{smallmatrix} \text{Ker}(\mathcal{O}_{V,W})^* \\ 0 \end{smallmatrix} \right]$ is clearly J -positive, it follows that $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$ is J -positive if and only if $\widehat{\mathbb{G}}_{T[*]}[+](\mathcal{M}_{\mathfrak{D}}^{[\perp]})_1$ is J -positive. Statement (2) follows in a similar way. \square

Lemma 4.5. (1) *The subspace $\mathbb{G}_{T[*]}$ is J -positive if and only if $I + TT^{[*]}$ is J -positive on the subspace $P_{H_{p+m}^2(\Pi_+)}\mathcal{M}_{\mathfrak{D}}^{[\perp]} = \text{Ran } J(\mathcal{C}_{Z,[X-Y]})^*$.*

(2) *The subspace $(\mathcal{M}_{\mathfrak{D}}^{[\perp]})_1$ is J -positive if and only if the subspace $P_{H_{p+m}^2(\Pi_-)}\mathcal{M}_{\mathfrak{D}} = \text{Ran } \mathcal{O}_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix} \right], W}$ is J -negative.*

(3) *The subspace $\widehat{\mathbb{G}}_T$ is J -negative if and only if $I + T^{[*]}T$ is a J -negative operator on the subspace $P_{H_{p+m}^2(\Pi_-)}\mathcal{M}_{\mathfrak{D}} = \text{Ran } \mathcal{O}_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix} \right], W}$.*

(4) *The subspace $\mathcal{M}_{\mathfrak{D},1}$ is J -negative if and only if the subspace $P_{H_{p+m}^2(\Pi_+)}\mathcal{M}_{\mathfrak{D}}^{[\perp]} = \text{Ran } J(\mathcal{C}_{Z,[X-Y]})^*$ is J -positive.*

Proof. To prove (1), note that $\mathbb{G}_{T^{[*]}}$ being a J -positive subspace means that

$$\left\langle \begin{bmatrix} I \\ T^{[*]} \end{bmatrix} x, \begin{bmatrix} I \\ T^{[*]} \end{bmatrix} x \right\rangle_{J \oplus J} = \langle (I + TT^{[*]})x, x \rangle_J \geq 0$$

for all $x \in \text{Ran } J(\mathcal{C}_{Z,[X-Y]})^*$, i.e., that $I + TT^{[*]}$ is a J -positive operator.

To prove (2), use (4.18) to see that elements \mathbf{g} of $(\mathcal{M}_{\mathfrak{D}}^{[\perp]})_1$ have the form

$$\mathbf{g} = \begin{bmatrix} \mathcal{O}_{V,W} \\ \mathcal{O}_{U,W}(\mathcal{G}_{U,W})^{-1}\mathcal{G}_{V,W} \end{bmatrix} x \quad \text{for some } x \in \mathbb{C}^{n_W}.$$

The associated J -gramian is then given by

$$\begin{aligned} & [(\mathcal{O}_{V,W})^* \quad \mathcal{G}_{V,W}(\mathcal{G}_{U,W})^{-1}(\mathcal{O}_{U,W})^*] \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} \mathcal{O}_{V,W} \\ \mathcal{O}_{U,W}(\mathcal{G}_{U,W})^{-1}\mathcal{G}_{V,W} \end{bmatrix} \\ &= \mathcal{G}_{V,W} - \mathcal{G}_{V,W}(\mathcal{G}_{U,W})^{-1}\mathcal{G}_{V,W}. \end{aligned}$$

By a Schur-complement analysis, this defines a negative semidefinite operator (in fact by our Nondegeneracy Assumption, a negative definite operator) if and only if

$$\begin{bmatrix} \mathcal{G}_{V,W} & \mathcal{G}_{V,W} \\ \mathcal{G}_{V,W} & \mathcal{G}_{U,W} \end{bmatrix} = \begin{bmatrix} \mathcal{G}_{V,W}^{\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I_{\text{Ran } \mathcal{G}_{V,W}} & \mathcal{G}_{V,W}^{\frac{1}{2}} \\ \mathcal{G}_{V,W}^{\frac{1}{2}} & \mathcal{G}_{U,W} \end{bmatrix} \begin{bmatrix} \mathcal{G}_{V,W}^{\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \prec 0,$$

which in turn happens if and only if

$$\begin{bmatrix} I_{\text{Ran } \mathcal{G}_{V,W}} & \mathcal{G}_{V,W}^{\frac{1}{2}} \\ \mathcal{G}_{V,W}^{\frac{1}{2}} & \mathcal{G}_{U,W} \end{bmatrix} \prec 0.$$

Yet another Schur-complement analysis converts this to the condition

$$\mathcal{G}_{U,W} - \mathcal{G}_{V,W} \prec 0$$

which is equivalent to $\text{Ran } \mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W}$ being a J -negative subspace.

The proofs of statements (3) and (4) are parallel to those of (1) and (2) respectively. \square

Lemma 4.6. *The Pick matrix $\mathbf{\Gamma}_{\mathfrak{D}}$ (1.18) can be factored as follows:*

$$\mathbf{\Gamma}_{\mathfrak{D}} = \begin{bmatrix} -\mathcal{C}_{Z,[X-Y]} & 0 \\ 0 & (\mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W})^* J \end{bmatrix} \begin{bmatrix} I & T \\ T^{[*]} & -I \end{bmatrix} \begin{bmatrix} -J(\mathcal{C}_{Z,[X-Y]})^* & 0 \\ 0 & \mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W} \end{bmatrix}. \quad (4.22)$$

Proof. Multiplying out the expression on the right-hand side in (4.22), we get

$$\begin{bmatrix} \mathcal{G}_{Z,[X-Y]}^J & -\mathcal{C}_{Z,[X-Y]} T \mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W} \\ -(\mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W})^* J T^{[*]} J (\mathcal{C}_{Z,[X-Y]})^* & -\mathcal{G}_{\begin{bmatrix} V \\ U \end{bmatrix}, W}^J \end{bmatrix},$$

which is exactly $\begin{bmatrix} \Gamma_L & \Gamma \\ \Gamma^* & \Gamma_R \end{bmatrix} =: \mathbf{\Gamma}_{\mathfrak{D}}$ as we can see from the identities (4.6) and (4.13). \square

Lemma 4.7. *The following conditions are equivalent:*

- (1) *The matrix $\mathbf{\Gamma}_{\mathfrak{D}}$ (1.18) is positive.*
- (2) *The subspace $P_{H^2_{p+m}(\Pi_-)}\mathcal{M}_{\mathfrak{D}} = \text{Ran } \mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W}$ is J -negative and the subspace $\mathbb{G}_{T^{[*]}}$ is J -positive.*
- (3) *The subspace $P_{H^2_{p+m}(\Pi_+)}\mathcal{M}_{\mathfrak{D}}^{[\perp]} = \text{Ran } J(\mathcal{C}_{Z, [X \ -Y]})^*$ is J -positive and the subspace $\widehat{\mathbb{G}}_T$ is J -negative.*

Proof. From the factorization (4.22) we see that $\mathbf{\Gamma}_{\mathfrak{D}} \succ 0$ if and only if the Hermitian form on the subspace $\begin{bmatrix} \text{Ran } J(\mathcal{C}_{Z, [X \ -Y]})^* \\ \text{Ran } \mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W} \end{bmatrix}$ induced by the operator $\begin{bmatrix} I & T \\ T^{[*]} & -I \end{bmatrix}$ in the $J \oplus J$ -inner product is positive. On the one hand we may consider the factorization

$$\begin{bmatrix} I & T \\ T^{[*]} & -I \end{bmatrix} = \begin{bmatrix} I & 0 \\ T^{[*]} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I - T^{[*]}T \end{bmatrix} \begin{bmatrix} I & T \\ 0 & I \end{bmatrix}$$

to deduce that $\begin{bmatrix} I & T \\ T^{[*]} & -I \end{bmatrix}$ is $(J \oplus J)$ -positive if and only if

- (i) the identity operator I is J -positive on $\text{Ran } J(\mathcal{C}_{Z, [X \ -Y]})^*$ (i.e., the subspace $\text{Ran } J(\mathcal{C}_{Z, [X \ -Y]})^*$ is J -positive), and
- (ii) $-I - T^{[*]}T$ is a J -positive operator on $\text{Ran } \mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W}$, i.e., $\widehat{\mathbb{G}}_T$ is a J -negative subspace.

Note that this analysis amounts to taking the J -symmetrized Schur complement of the matrix $\begin{bmatrix} I & T \\ T^{[*]} & -I \end{bmatrix}$ with respect to the (1,1)-entry. This establishes the equivalence of (1) and (3).

On the other hand we may take the J -symmetrized Schur complement of $\begin{bmatrix} I & T \\ T^{[*]} & -I \end{bmatrix}$ with respect to the (2,2)-entry, corresponding to the factorization

$$\begin{bmatrix} I & T \\ T^{[*]} & -I \end{bmatrix} = \begin{bmatrix} I & -T \\ 0 & I \end{bmatrix} \begin{bmatrix} I + TT^{[*]} & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ -T^{[*]} & I \end{bmatrix}.$$

In this way we see that $(J \oplus J)$ -positivity of $\begin{bmatrix} I & T \\ T^{[*]} & -I \end{bmatrix}$ corresponds to

- (i') $I + TT^{[*]}$ is a J -positive operator (i.e., the subspace $\mathbb{G}_{T^{[*]}}$ is J -positive), and
- (ii') minus the identity operator $-I$ is J positive on $\text{Ran } \mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W}$ (i.e., the subspace is $\text{Ran } \mathcal{O}_{\begin{bmatrix} V \\ U \end{bmatrix}, W}$ is J -negative).

This establishes the equivalence of (1) and (2). \square

To conclude the proof of Corollary 4.1 for the general BiTangential case (at least with the Nondegeneracy Assumption in place), it remains only to assemble the various pieces. By Lemma 4.4 part (1), we see that $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$

being J -positive is equivalent to

$$\mathbb{G}_{T[*]} \text{ and } (\mathcal{M}_{\mathfrak{D}}^{[\perp]})_1 \text{ are } J\text{-positive subspaces.} \quad (4.23)$$

By Lemma 4.5, we see that $(\mathcal{M}_{\mathfrak{D}}^{[\perp]})_1$ being J -positive is equivalent to $\text{Ran } \mathcal{O}_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right], W}$ being J -negative. We therefore may amend (4.23) to

$$\mathbb{G}_{T[*]} \text{ is } J\text{-positive and } P_{H_{p+m}^2(\Pi_+)} \mathcal{M}_{\mathfrak{D}} = \text{Ran } \mathcal{O}_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right], W} \text{ is } J\text{-negative} \quad (4.24)$$

which is exactly statement (2) in Lemma 4.7. Thus $(1) \Leftrightarrow (2)$ in Corollary 4.1 follows from $(1) \Leftrightarrow (2)$ in Lemma 4.7.

For the general BTOA case, the reproducing kernel space $\mathcal{H}(K_{\Theta, J})$ again can be identified with a range space, namely

$$\mathcal{H}(K_{\Theta, J}) = \text{Ran}(P_{H_{p+m}^2(\Pi_+)}^J - P_{\mathcal{M}_{\mathfrak{D}}}^J) \quad (4.25)$$

with lifted indefinite inner product, where $P_{H_{p+m}^2(\Pi_+)}^J$ and $P_{\mathcal{M}_{\mathfrak{D}}}^J$ are the J -orthogonal projections of $L_{p+m}^2(i\mathbb{R})$ onto $H_{p+m}^2(\Pi_+)$ and $\mathcal{M}_{\mathfrak{D}}$ respectively (see [12, Theorem 3.3]). Due to J -orthogonal decompositions

$$\begin{aligned} H_{p+m}^2(\Pi_+) &= \text{Ran } J(\mathcal{C}_{Z, [X \ -Y]})^* [+] \mathcal{M}_{\mathfrak{D}, 1} [+] \mathcal{M}_{\mathfrak{D}, 2}, \\ \mathcal{M} &= \widehat{\mathbb{G}}_T [+] \mathcal{M}_{\mathfrak{D}, 1} [+] \mathcal{M}_{\mathfrak{D}, 2}, \end{aligned}$$

we can simplify the difference of J -orthogonal projections to

$$P_{H_{p+m}^2}^J - P_{\mathcal{M}_{\mathfrak{D}}}^J = P_{\text{Ran } J(\mathcal{C}_{Z, [X \ -Y]})^*}^J - P_{\widehat{\mathbb{G}}_T}^J.$$

By a calculation as in the proof for Case 1, one can show that

$$\mathcal{H}(K_{\Theta, J}) = (\text{Ran } J(\mathcal{C}_{Z, [X \ -Y]})^*)_J [+] (\widehat{\mathbb{G}}_T)_{-J} \quad (4.26)$$

with the identity map a Kreĭn-space isomorphism, where the subscripts on the right hand side indicating that one should use the J -inner product for the first component but the $-J$ -inner product for the second component. We conclude that $\mathcal{H}(K_{\Theta, J})$ is a Hilbert space exactly when condition (3) in Lemma 4.7 holds. We now see that $(1) \Leftrightarrow (3)$ in Corollary 4.1 is an immediate consequence of $(1) \Leftrightarrow (3)$ in Lemma 4.7. \square

The above analysis actually establishes a bit more which we collect in the following Corollary.

Corollary 4.8. *The following conditions are equivalent:*

- (1) *The subspace $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$ is J -positive.*
- (2) *The subspace $(\mathcal{M}_{\mathfrak{D}}^{[\perp]})^{[\perp \mathcal{K}]}$ is J -negative.*

Proof. We have seen in Lemma 4.4 part (2) that $(\mathcal{M}_{\mathfrak{D}}^{[\perp]})^{[\perp \mathcal{K}]}$ being J -negative is equivalent to

$$\widehat{\mathfrak{G}}_T \text{ and } (\mathcal{M}_{\mathfrak{D}}^{[\perp]})_1 \text{ are } J\text{-negative subspaces.} \quad (4.27)$$

Lemma 4.5 (4) tells us that $\mathcal{M}_{\mathfrak{D},1}$ being J -negative is equivalent to $\text{Ran } J(\mathcal{C}_{Z,[X-Y]})^*$ being J -positive. Thus condition (4.27) can be amended to

$$\widehat{\mathbb{G}}_T \text{ is negative and } P_{H^2_{p+m}(\Pi_+)} \mathcal{M}_{\mathfrak{D}}^{[\perp]} = \text{Ran } J(\mathcal{C}_{Z,[X-Y]})^* \text{ is } J\text{-positive.} \quad (4.28)$$

We next use the equivalence of (1) \Leftrightarrow (3) in Theorem 4.7 to see that condition (4.28) is also equivalent to $\mathbf{\Gamma}_{\mathfrak{D}} \succ 0$. We then use the equivalence (1) \Leftrightarrow (2) in Theorem 4.7 to see that this last condition in turn is equivalent to $\mathcal{M}_{\mathfrak{D}}^{[\perp, \mathcal{K}]}$ being J -positive. \square

5. INTERPOLATION PROBLEMS IN THE GENERALIZED SCHUR CLASS

Much of the previous analysis extends from the Schur class $\mathcal{S}^{p \times m}(\Pi_+)$ to a larger class $\mathcal{S}_{\kappa}^{p \times m}(\Pi_+)$ (generalized Schur class) consisting of $\mathbb{C}^{p \times m}$ -valued functions that are meromorphic on Π_+ with total pole multiplicity equal κ and such that their L^∞ norm (that is, $\sup_{y \in \mathbb{R}} \|S(iy)\|$) does not exceed one. The values $S(iy)$ are understood in the sense of non-tangential boundary limits that exist for almost all $y \in \mathbb{R}$. The multiplicity of a pole z_0 for a matrix-valued function S is defined as the sum of absolute values of all negative partial multiplicities appearing in the Smith form of S at z_0 (see e.g. [9, Theorem 3.1.1]). Then the total pole multiplicity of S is defined as the sum of multiplicities of all poles. Let us introduce the notation

$$m_P(S) = \text{sum of all pole multiplicities of } S \text{ over all poles in } \Pi_+.$$

It follows by the maximum modulus principle that $\mathcal{S}_0^{p \times m}(\Pi_+)$ is just the classical Schur class. Generalized Schur functions appeared first in [40] in the interpolation context and were comprehensively studied by Kreĭn and Langer in [33, 34]. Later work on the classes \mathcal{S}_{κ} include [20], [29], [18], and [1], as well as [27], [37], [10], [11] and the book [9] in the context of interpolation.

The class $\mathcal{S}_{\kappa}(\Pi_+)$ can alternatively be characterized by any of the following conditions:

- (1) $\text{sq}_-(K_S) = \kappa$ where the kernel K_S is given by (2.5).
- (2) $\text{sq}_-(\mathbf{K}_S) = \kappa$, where \mathbf{K}_S is the 2×2 -block matrix kernel (2.6).
- (3) S admits left and right (coprime) *Kreĭn-Langer factorizations*

$$F(\lambda) = S_R(\lambda) \vartheta_R(\lambda)^{-1} = \vartheta_L(\lambda)^{-1} S_L(\lambda),$$

where $S_L, S_R \in \mathcal{S}^{p \times m}(\Pi_+)$ and ϑ_L and ϑ_R are matrix-valued finite Blaschke products of degree κ (see [34] for the scalar-valued case and [20] for the Hilbert-space operator-valued case). By a $\mathbb{C}^{n \times n}$ -valued finite Blaschke product we mean the product of κ Blaschke (or Blaschke-Potapov) factors

$$I_n - P + \frac{\lambda - \alpha}{\lambda + \bar{\alpha}} P$$

where $\alpha \in \Pi_+$ and P is an orthogonal projection in \mathbb{C}^n .

There is also an intrinsic characterization of matrix triples (C, A, B) which can arise as the pole triple over the unit disk for a generalized Schur class function—see [17] for details.

Let us take another look at the BiTangential Nevanlinna-Pick problem (1.2)–(1.4). If the Pick matrix (1.6) is not positive semidefinite, the problem has no solutions in the Schur class $\mathcal{S}^{p \times n}(\Pi_+)$, by Theorem 1.1. However, there always exist generalized Schur functions that are analytic at all interpolation nodes z_i, w_j and satisfy interpolation conditions (1.2)–(1.4). One can show that there exist such functions with only one pole of a sufficiently high multiplicity at any preassigned point in Π_+ . The question of interest is to find the smallest integer κ , for which interpolation conditions (1.2)–(1.4) are met for some function $S \in \mathcal{S}_\kappa^{p \times m}(\Pi_+)$ and then to describe the set of all such functions.

The same question makes sense in the more general setting of the **BTOA-NP** interpolation problem: *given a Π_+ -admissible BTOA interpolation data set (1.14), find the smallest integer κ , for which interpolation conditions (1.11)–(1.13) are satisfied for some function $S \in \mathcal{S}_\kappa^{p \times m}(\Pi_+)$ which is analytic on $\sigma(Z) \cup \sigma(W)$, and describe the set of all such functions.*

The next theorem gives the answer to the question above in the so-called nondegenerate case.

Theorem 5.1. *Suppose that $\mathfrak{D} = (X, Y, Z; U, V, W; \Gamma)$ is a Π_+ -admissible BTOA interpolation data set and let us assume that the BTOA-Pick matrix $\Gamma_{\mathfrak{D}}$ defined by (1.18) is invertible. Let κ be the smallest integer for which there is a function $S \in \mathcal{S}_\kappa^{p \times m}(\Pi_+)$ which is analytic on $\sigma(W) \cup \sigma(Z)$ and satisfies the interpolation conditions (1.11)–(1.13). Then κ is given by any one of the following three equivalent formulas:*

- (1) $\kappa = \nu_-(\Gamma_{\mathfrak{D}})$, the number of negative eigenvalues of $\Gamma_{\mathfrak{D}}$.
- (2) $\kappa = \nu_-(\mathcal{M}_{\mathfrak{D}}^{[\perp]\mathcal{K}})$, the negative signature of the Kreĭn-space $\mathcal{M}_{\mathfrak{D}}^{[\perp]\mathcal{K}}$ in the J -inner product.
- (3) $\kappa = \nu_-(\mathcal{H}(K_{\Theta, J}))$, the negative signature of the reproducing kernel Pontryagin space $\mathcal{H}(K_{\Theta, J})$, where Θ is defined as in (2.27) and $K_{\Theta, J}$ as in (2.20).

Furthermore, the function S belongs to the generalized Schur class $\mathcal{S}_\kappa^{p \times m}(\Pi_+)$ and satisfies the interpolation conditions (1.11)–(1.13) if and only if it is of the form

$$S(\lambda) = (\Theta_{11}(\lambda)G(\lambda) + \Theta_{12}(\lambda))(\Theta_{21}(\lambda)G(\lambda) + \Theta_{22}(\lambda))^{-1} \quad (5.1)$$

for a Schur class function $G \in \mathcal{S}^{p \times m}(\Pi_+)$ such that

$$\det(\psi(\lambda)(\Theta_{21}(\lambda)G(\lambda) + \Theta_{22}(\lambda))) \neq 0, \quad \lambda \in \Pi_+ \setminus (\sigma(Z) \cup \sigma(W)) \quad (5.2)$$

where ψ is the $m \times m$ -matrix function defined in (2.27).

5.1. The state-space approach. The direct proof of the necessity of condition (1) in Theorem 5.1 for the existence of class- $\mathcal{S}_\kappa^{p \times m}(\Pi_+)$ solution of the interpolation conditions (1.11)–(1.13) relies on the characterization of the class $\mathcal{S}_\kappa^{p \times m}(\Pi_+)$ in terms of the kernel (2.6) mentioned above: *a $\mathbb{C}^{p \times n}$ -valued function meromorphic on Π_+ belongs to $\mathcal{S}_\kappa^{p \times m}(\Pi_+)$ if and only if the kernel $\mathbf{K}_S(\lambda, \lambda_*; \zeta, \zeta_*)$ defined as in (2.6) has κ negative squares on Ω_S^4 :*

$$\text{sq}_- \mathbf{K}_S = \kappa, \quad (5.3)$$

where $\Omega_S \subset \Pi_+$ is the domain of analyticity of S . The latter equality means that the block matrix $[\mathbf{K}_S(z_i, z_i; z_j, z_j)]_{i,j=1}^N$ has at most κ negative eigenvalues for any choice of finitely many points $z_1, \dots, z_N \in \Omega_S$, and it has exactly κ negative eigenvalues for at least one such choice.

Now suppose that $S \in \mathcal{S}_\kappa^{p \times m}(\Pi_+)$ satisfies the interpolation conditions (1.11)–(1.13). The kernel \mathbf{K}_S satisfying condition (5.3) still admits the Kolmogorov decomposition (2.8), but this time the state space \mathcal{X} is a Pontryagin space of negative index κ . All computations following formula (2.8) go through with Π_+ replaced by Ω_S showing that the matrix $\mathbf{\Gamma}'_{\mathfrak{D}}$ defined in (2.12) is equal to the Pick matrix $\mathbf{\Gamma}_{\mathfrak{D}}$ given in (1.18). Note that the operations bringing the kernel \mathbf{K}_S to the matrix $\mathbf{\Gamma}_{\mathfrak{D}}$ amount to a sophisticated conjugation of the kernel \mathbf{K}_S . We conclude that $\nu_-(\mathbf{\Gamma}_{\mathfrak{D}}) = \nu_-(\mathbf{\Gamma}'_{\mathfrak{D}}) \leq \kappa$. Once one of the sufficiency arguments has been carried out (by whatever method) to show that $\nu_-(\mathbf{\Gamma}_{\mathfrak{D}}) = \nu_-(\mathbf{\Gamma}'_{\mathfrak{D}}) < \kappa$ implies that there is a function S in a generalized Schur class $\mathcal{S}_{\kappa'}^{p \times m}(\Pi_+)$ with $\kappa' < \kappa$ satisfying the interpolation conditions, then $\nu_-(\mathbf{\Gamma}_{\mathfrak{D}}) < \kappa$ leads to a contradiction to the minimality property of κ . We conclude that $\nu_-(\mathbf{\Gamma}_{\mathfrak{D}}) = \kappa$ is necessary for κ to be the smallest integer so that there is a solution S of class $\mathcal{S}_\kappa^{p \times m}(\Pi_+)$ of the interpolation conditions (1.11)–(1.13).

We now suppose that $\nu_-(\mathbf{\Gamma}_{\mathfrak{D}}) = \kappa$. The identity (2.20) relies on equality (2.19) and on the assumption that $\mathbf{\Gamma}_{\mathfrak{D}}$ is invertible. In particular, the matrix $\Theta(\lambda)$ still is J -unitary for each $\lambda \in i\mathbb{R}$, i.e., equalities (2.21) hold for all $\lambda \in i\mathbb{R}$. By using the controllability/observability assumptions on (Z, X) and (U, W) , it follows from the formula on the right hand side of (2.20) that the kernel $K_{\Theta, J}$ (2.20) has κ negative squares on $\Omega_\Theta = \Pi_+ \setminus \sigma(W)$ (the points of analyticity for Θ in the right half plane Π_+):

$$\text{sq}_- K_{\Theta, J} = \kappa.$$

We shall have need of the *Potapov-Ginsburg transform* $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$ of a given block 2×2 -block matrix function $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{21} \\ \Theta_{21}^* & \Theta_{22} \end{bmatrix}$ (called the *Redheffer transform* in [9]) defined by

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} := \begin{bmatrix} \Theta_{12}\Theta_{22}^{-1} & \Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{21} \\ \Theta_{22}^{-1} & -\Theta_{22}^{-1}\Theta_{21} \end{bmatrix}.$$

This transform is the result of rearranging the inputs and outputs in the system of equations

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ x_1 \end{bmatrix} \quad (5.4)$$

to have the form

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad (5.5)$$

and in circuit theory has the interpretation as the change of variable from the *chain formalism* (5.4) to the *scattering formalism* (5.5). Based on this connection it is not hard to show that

$$\text{sq}_- K_U = \text{sq}_- K_{\Theta, J} = \kappa$$

where the notation K_U is as in (2.5) and $K_{\Theta, J}$ as in (2.20) (see [9, Theorem 13.1.3]). We conclude that U is in the generalized Schur class $\mathcal{S}_\kappa^{(p+m) \times (m+p)}(\Pi_+)$. By the Kreĭn-Langer factorization result for the generalized Schur class (see [34]), it follows that κ is also equal to the total pole multiplicity of U over points in Π_+ :

$$m_P(U) = \kappa.$$

We would like to show next that

$$m_P(U_{22}) = m_P(\Theta_{22}^{-1} \Theta_{21}) = \kappa. \quad (5.6)$$

Verification of this formula will take several steps and follow the analysis in [9, Chapter 13]. We first note that the calculations (2.25)–(2.29) go through unchanged so we still have the Beurling-Lax representation

$$\mathcal{M}_{\mathfrak{D}} = \Theta \cdot H_{p+m}^2(\Pi_+) \quad (5.7)$$

where $\mathcal{M}_{\mathfrak{D}}$ also has the representation (2.25). The observability assumption on the output pair (U, W) translates to an additional structural property on $\mathcal{M}_{\mathfrak{D}}$:

- (U, W) observable implies

$$\mathcal{M}_{\mathfrak{D}} \cap \begin{bmatrix} L^2(i\mathbb{R}) \\ 0 \end{bmatrix} = \mathcal{M}_{\mathfrak{D}} \cap \begin{bmatrix} H_p^2(\Pi_+) \\ 0 \end{bmatrix}. \quad (5.8)$$

Making use of (5.7), condition (5.8) translates to an explicit property of Θ , namely:

$$f \in H_p^2(\Pi_+), g \in H_m^2(\Pi_+), \Theta_{21}f + \Theta_{22}g = 0 \Rightarrow \Theta_{11}f + \Theta_{12}g \in H_p^2(\Pi_+).$$

Solving the first equation for g gives $g = -\Theta_{22}^{-1}\Theta_{21}f$ and this last condition can be rewritten as

$$f \in H_p^2(\Pi_+), \Theta_{22}^{-1}\Theta_{21}f \in H_m^2(\Pi_+) \Rightarrow (\Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{21})f \in H_p^2(\Pi_+),$$

or, more succinctly,

$$f \in H_p^2(\Pi_+), U_{22}f \in H_m^2(\Pi_+) \Rightarrow U_{12}f \in H_p^2(\Pi_+).$$

This last condition translates to

$$m_P(U_{22}) = m_P \left(\begin{bmatrix} U_{12} \\ U_{22} \end{bmatrix} \right). \quad (5.9)$$

Similarly, the controllability assumption on the input pair (Z, X) translates to an additional structural property on $\mathcal{M}_{\mathfrak{D}}$, namely:

- (Z, X) controllable implies

$$P_{\begin{bmatrix} 0 \\ H_m^2(\Pi_+) \end{bmatrix}} (\mathcal{M}_{\mathfrak{D}} \cap H_{p+m}^2(\Pi_+)) = \begin{bmatrix} 0 \\ H_m^2(\Pi_+) \end{bmatrix}. \quad (5.10)$$

In terms of Θ , from the representation (5.7) we see that this means that, given any $h \in H_m^2(\Pi_+)$, we can find $f \in H_p^2(\Pi_+)$ and $g \in H_m^2(\Pi_+)$ so that

$$\Theta_{11}f + \Theta_{12}g \in H_p^2(\Pi_+), \quad \Theta_{21}f + \Theta_{22}g = h.$$

We can solve the second equation for g

$$g = \Theta_{22}^{-1}h - \Theta_{22}^{-1}\Theta_{21}f \in H_m^2(\Pi_+)$$

and rewrite the first expression in terms of f and h :

$$(\Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{21})f + \Theta_{12}\Theta_{22}^{-1}h \in H_p^2(\Pi_+).$$

Putting the pieces together, we see that an equivalent form of condition (5.10) is: *for any $h \in H_m^2(\Pi_+)$, there exists an $f \in H_p^2(\Pi_+)$ such that*

$$\Theta_{22}^{-1}h - \Theta_{22}^{-1}\Theta_{21}f \in H_m^2(\Pi_+), \quad (\Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{21})f + \Theta_{12}\Theta_{22}^{-1}h \in H_p^2(\Pi_+).$$

More succinctly,

$$\begin{aligned} h \in H_m^2(\Pi_+) &\Rightarrow \exists f \in H_p^2(\Pi_+) \text{ so that} \\ U_{21}h + U_{22}f &\in H_m^2(\Pi_+), \quad U_{12}f + U_{11}h \in H_p^2(\Pi_+), \end{aligned}$$

or, in column form, for each $h \in H_m^2(\Pi_+)$ there exists $f \in H_p^2(\Pi_+)$ so that

$$\begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} h + \begin{bmatrix} U_{12} \\ U_{22} \end{bmatrix} f \in H_{p+m}^2(\Pi_+).$$

The meaning of this last condition is:

$$m_P(U) = m_P \left(\begin{bmatrix} U_{12} \\ U_{22} \end{bmatrix} \right). \quad (5.11)$$

Combining (5.9) with (5.11) gives us (5.6) as wanted.

Since Θ is not J -contractive in Π_+ anymore, we cannot conclude that $\Theta_{22}^{-1}\Theta_{21}$ is contraction valued. However, due to equalities (2.21), the function $\Theta_{22}(\lambda)^{-1}\Theta_{21}(\lambda)$ is a contraction for each $\lambda \in i\mathbb{R}$. Therefore, $\Theta_{22}^{-1}\Theta_{21}$ belongs to the generalized Schur class $\mathcal{S}_{\kappa}^{m \times p}(\Pi_+)$. We next wish to argue that

$$\text{wno det } \Theta_{22} + \text{wno det } \psi = \kappa, \quad (5.12)$$

where ψ is given by (2.27). From the representation (5.7) and the form of $\mathcal{M}_{\mathfrak{D}}$ in (5.7) we see that

$$\Theta_{22} \begin{bmatrix} \Theta_{22}^{-1}\Theta_{21} & I_m \end{bmatrix} H_{p+m}^2(\Pi_+) = \begin{bmatrix} \Theta_{21} & \Theta_{22} \end{bmatrix} H_{p+m}^2(\Pi_+) = \psi^{-1} H_m^2(\Pi_+).$$

We rewrite this equality as

$$\begin{bmatrix} \Theta_{22}^{-1}\Theta_{21} & I_m \end{bmatrix} H_{p+m}^2(\Pi_+) = \Theta_{22}^{-1}\psi^{-1}H_m^2(\Pi_+). \quad (5.13)$$

In particular,

$$\Theta_{22}^{-1}\psi^{-1}H_m^2(\Pi_+) \supset H_m^2(\Pi_+)$$

so the matrix function $\Theta_{22}^{-1}\psi^{-1}$ has no zeros (in the sense of its Smith-McMillan form) in Π_+ . As Θ_{22}^{-1} and ψ^{-1} are invertible on the boundary $i\mathbb{R}$, we see that $\text{wno det}(\Theta_{22}^{-1}\psi^{-1})$ is well-defined and by the Argument Principle we have

$$\begin{aligned} -\text{wno det } \Theta_{22} - \text{wno det }(\psi) &= \text{wno det}(\Theta_{22}^{-1}\psi^{-1}) \\ &= m_Z(\det(\Theta_{22}^{-1}\psi^{-1})) - m_P(\det(\Theta_{22}^{-1}\psi^{-1})) \\ &= -m_P(\det(\Theta_{22}^{-1}\psi^{-1})) = -m_P(\Theta_{22}^{-1}\psi^{-1}) \\ &= -\dim P_{H_m^2(\Pi_-)} \Theta_{22}^{-1}\psi^{-1}H_m^2(\Pi_+) \end{aligned} \quad (5.14)$$

where $m_Z(S)$ is the total zero multiplicity of the rational matrix function S over all zeros in Π_+ . On the other hand we have

$$\dim P_{H_m^2(\Pi_-)} \begin{bmatrix} \Theta_{22}^{-1}\Theta_{21} & I_m \end{bmatrix} H_{p+m}^2(\Pi_+) = m_P(\Theta_{22}^{-1}\Theta_{21}) = \kappa \quad (5.15)$$

where we make use of (5.6) for the last step. Combining (5.14) and (5.15) with (5.13) finally brings us to (5.12).

In addition to the Beurling-Lax representation (2.29) or (5.7), we also still have the Beurling=Lax representation (2.26) for $\mathcal{M}_{\mathfrak{D},-}$ with ψ, ψ^{-1} given by (2.27) and (2.28). However, the condition (2.33) should be modified as follows:

- A meromorphic function $S: \Pi_+ \rightarrow \mathbb{C}^{p \times m}$ has total pole multiplicity at most κ over Π_+ and satisfies the interpolation conditions (1.11)–(1.13) if and only if there is an $m \times m$ -matrix valued function Ψ analytic on Π_+ with $\det \Psi$ having no zeros on $\sigma(Z) \cup \sigma(W)$ and with κ zeros in Π_+ such that

$$\begin{bmatrix} S \\ I_m \end{bmatrix} \psi^{-1}\Psi H_m^2(\Pi_+) \subset \mathcal{M}_{\mathfrak{D}}. \quad (5.16)$$

Now instead of (2.34), we have

$$\begin{bmatrix} S \\ I_m \end{bmatrix} \psi^{-1}\Psi = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \quad (5.17)$$

for some $(p+m) \times m$ matrix function $\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \in H_{(p+m) \times m}^2(\Pi_+)$. Then we conclude from the J -unitarity of Θ on $i\mathbb{R}$ (exactly as in Section 2) that for almost all $\lambda \in i\mathbb{R}$, the matrix $Q_2(\lambda)$ is invertible whereas the matrix $G(\lambda) = Q_1(\lambda)Q_2(\lambda)^{-1}$ is a contraction. The identity (2.35) arising from looking at the bottom component of (5.17) must be modified to read

$$\psi^{-1}\Psi = \Theta_{21}Q_1 + \Theta_{22}Q_2 = \Theta_{22}(\Theta_{22}^{-1}\Theta_{21}G + I_m)Q_2$$

leading to the modification of (2.36):

$$\begin{aligned} \text{wno det } \psi^{-1} + \text{wno det } \Psi = \\ \text{wno det } \Theta_{22} + \text{wno det}(\Theta_{22}^{-1}\Theta_{21}G + I_m) + \text{wno det } Q_2. \end{aligned}$$

The identity (2.37) must be replaced by (5.12). Using that $\text{wno det } \Psi = \kappa$, with all these adjustments in place we still arrive at $\text{wno det } Q_2 = 0$ and hence Q_2 has no zeros in Π_+ and G extends inside Π_+ as a Schur-class function. The representation (5.1) follows from (5.17) as well as the equality $\Psi = \psi(\Theta_{21}G + \Theta_{22})Q_2$. Since Ψ has no zeros in $\sigma(Z) \cap \sigma(W)$ while $\psi(\Theta_{21}G + \Theta_{22})$ and Q_2 are analytic on all of Π_+ , we see that $\psi(\Theta_{21}G + \Theta_{22})$ has no zeros in $\sigma(Z) \cap \sigma(W)$ as well.

Conversely, for any $G \in \mathcal{S}^{p \times m}(\Pi_+)$ such that $\psi(\Theta_{21}G + \Theta_{22})$ has no zeros on $\sigma(Z) \cup \sigma(W)$, we let

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \Theta \begin{bmatrix} G \\ I_m \end{bmatrix}, \quad \Psi = \psi S_2, \quad S = S_1 S_2^{-1},$$

so that

$$\begin{bmatrix} S \\ I_m \end{bmatrix} \psi^{-1} \Psi = \Theta \begin{bmatrix} G \\ I_m \end{bmatrix}.$$

Since Θ is J -unitary on $i\mathbb{R}$ and G is a Schur-class, it follows that $S(\lambda)$ is contractive for almost all $\lambda \in i\mathbb{R}$. Since $\det \Psi$ has no zeros on $\sigma(Z) \cup \sigma(W)$ and has κ zeros in Π_+ , due to the equalities

$$\text{wno det } \Psi = \text{wno det } \psi + \text{wno det } \Theta_{22} + \text{wno det}(\Theta_{22}^{-1}\Theta_{21}G + I) = \kappa$$

we see that S satisfies the interpolation conditions (1.11)–(1.13) by the criterion (5.16) and has total pole multiplicity at most κ in Π_+ . However, since $\nu_-(\Gamma_{\mathfrak{D}}) = \kappa$, by the part of the sufficiency criterion already proved we know that S must have at least κ poles in Π_+ . Thus S has exactly κ poles in Π_+ and therefore is in the $\mathcal{S}_\kappa^{p \times m}(\Pi_+)$ -class.

5.2. The Fundamental Matrix Inequality approach for the generalized Schur-class setting. The Fundamental Matrix Inequality method extends to the present setting as follows. As in the definite case, we extend the interpolation data by an arbitrary finite set of additional full-matrix-value interpolation conditions to conclude that the kernel $\Gamma_{\mathfrak{D}}(z, \zeta)$ defined as in (2.51) has at most κ negative squares in $\Omega_S \setminus \sigma(W)$. Since the constant block (the matrix $\Gamma_{\mathfrak{D}}$) has κ negative eigenvalues (counted with multiplicities), it follows that $\text{sp}_- \Gamma_{\mathfrak{D}}(z, \zeta) = \kappa$ which holds if and only if the Schur complement of $\Gamma_{\mathfrak{D}}$ in (2.51) is a positive kernel on $\Omega_S \setminus \sigma(W)$:

$$\frac{I_p - S(z)S(\zeta)^*}{z + \bar{\zeta}} - [I_p \quad -S(z)] \mathbf{C}(zI - \mathbf{A})^{-1} \Gamma_{\mathfrak{D}}^{-1} (\bar{\zeta}I - \mathbf{A}^*)^{-1} \mathbf{C}^* \begin{bmatrix} I_p \\ -S(\zeta)^* \end{bmatrix} \succeq 0.$$

As in Section 2.1, the latter positivity condition can be written in the form (2.52) (all we need is formula (2.20) which still holds true) and eventually, implies equality (2.54) for some $G \in \mathcal{S}^{p \times m}(\Pi_+)$, which in turn, implies the

representation (5.1). However, establishing the necessity of the condition (5.2) requires a good portion of extra work. Most of the known proofs are still based the Argument Principle (the winding number computations [9] or the operator-valued version of Rouché's theorem [26]). For example, it can be shown that if K is a $p \times m$ matrix-valued polynomial satisfying interpolation conditions (1.11)–(1.13) and if φ is the inner function given (analogously to (2.27)) by

$$\varphi(z) = I_p - X^*(zI + Z^*)^{-1}\tilde{P}^{-1}X,$$

where the positive definite matrix \tilde{P} is uniquely defined from the Lyapunov equation $\tilde{P}Z + Z^*\tilde{P} = XX^*$, then the matrix function

$$\Sigma := \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} \varphi^{-1} & -\varphi^{-1}K \\ 0 & \psi \end{bmatrix} \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \quad (5.18)$$

is analytic on Π_+ . Let us observe that by the formulas (2.17), (2.19) and well known properties of determinants,

$$\begin{aligned} \det \Theta(\lambda) &= \det (I - \mathbf{C}(\lambda I - \mathbf{A})^{-1}\Gamma_{\mathfrak{D}}^{-1}\mathbf{C}J) \\ &= \det (I - \mathbf{C}J\mathbf{C}(\lambda I - \mathbf{A})^{-1}\Gamma_{\mathfrak{D}}^{-1}) \\ &= \det(\Gamma_{\mathfrak{D}}(\lambda I - \mathbf{A}) + \Gamma_{\mathfrak{D}}\mathbf{A} + \mathbf{A}^*\Gamma_{\mathfrak{D}}) \cdot \det((\lambda I - \mathbf{A})^{-1}\Gamma_{\mathfrak{D}}^{-1}) \\ &= \frac{\det(\lambda I + \mathbf{A}^*)}{\det(\lambda I - \mathbf{A})} = \frac{\det(\lambda I - Z)\det(\lambda I + W^*)}{\det(\lambda I + Z^*)\det(\lambda I - W)}. \end{aligned}$$

Similar computations show that

$$\det \psi(\lambda) = \frac{\det(\lambda I - W)}{\det(\lambda I + W^*)}, \quad \det \varphi(z) = \frac{\det(\lambda I - Z)}{\det(\lambda I + Z^*)}.$$

Combining the three latter equalities with (5.18) gives $\det \Sigma(\lambda) \equiv 1 \neq 0$. Therefore, for $G \in \mathcal{S}^{p \times m}$, the total pole multiplicity of the function

$$\Upsilon = (\Sigma_{11}G + \Sigma_{12})(\Sigma_{21}G + \Sigma_{22})^{-1}$$

is the same as the number of zeros of the denominator

$$\Sigma_{21}G + \Sigma_{22} = \psi(\Theta_{21}G + \Theta_{22}),$$

that is κ , by the winding number argument. On the other hand, since

$$S = K + \varphi\Upsilon\psi, \quad (5.19)$$

as can be seen from (5.1) and (5.18), the total pole multiplicity of S equals κ if no poles of Υ occur at zeros of φ and Ψ , that is, in $\sigma(Z) \cup \sigma(W)$. We note that the form (5.19) where K, φ, ψ are part of the data and Υ is a free meromorphic function with no poles on $i\mathbb{R}$ but κ poles in Π_+ (including possibly at points of $\sigma(W) \cup \sigma(Z)$) corresponds to a variant of the interpolation problem (1.11), (1.12), (1.13) sometimes called the *Takagi-Sarason problem* (see [9, Chapter 19], [16]). It turns out that discarding the side-condition (5.2) on the Schur-class free-parameter function G leads to a parametrization of the set of all solutions of the Takagi-Sarason problem.

5.3. Indefinite kernels and reproducing kernel Pontryagin spaces.

From the formula (2.20) for $K_{\Theta,J}$, we see from the observability assumption on (\mathbf{C}, \mathbf{A}) (equivalently, the observability and controllability assumptions on $(\begin{bmatrix} V \\ U \end{bmatrix}, W)$ and $(Z, \begin{bmatrix} X & -Y \end{bmatrix})$) that

$$\nu_-(\Gamma_{\mathfrak{D}}) = \text{sq}_-(K_{\Theta,J}).$$

By the general theory of reproducing kernel Hilbert spaces sketched in Section 5.3, it follows that $\mathcal{H}(K_{\Theta,J})$ is a Pontryagin space with negative index $\nu_-(\mathcal{H}(K_{\Theta,J}))$ equal to the number of negative eigenvalues of $\Gamma_{\mathfrak{D}}$:

$$\nu_-(\mathcal{H}(K_{\Theta,J})) = \nu_-(\Gamma_{\mathfrak{D}}).$$

We conclude that the formula for κ in statement (1) agrees with that in statement (2) in Theorem 5.1.

5.4. The Grassmannian/Kreĭn-space approach for the generalized Schur-class setting. The Grassmannian approach extends to the present setting as follows. The suitable analog of Lemma 3.2 is the following:

Lemma 5.2. *Suppose that \mathcal{M} is a closed subspace of a Kreĭn-space \mathcal{K} such that the \mathcal{K} -relative orthogonal complement $\mathcal{M}^{[\perp]}$ has negative signature equal κ . If \mathcal{G} is a negative subspace of \mathcal{M} , then \mathcal{G} has codimension at least κ in any maximal negative subspace of \mathcal{K} . Moreover, the codimension of such a \mathcal{G} in any maximal negative subspace of \mathcal{K} is equal to κ if and only if \mathcal{G} is a maximal negative subspace of \mathcal{M} .*

Let us now assume that we are given a Π_+ -admissible interpolation data set \mathfrak{D} with $\Gamma_{\mathfrak{D}}$ invertible. Then $\mathcal{M}_{\mathfrak{D}}$ given by (2.25) is a regular subspace of the Kreĭn space $L^2_{p+m}(i\mathbb{R})$ with the $J(= \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix})$ -inner product.

With Lemma 5.2 in hand, we argue that $\nu_-(\mathcal{M}_{\mathfrak{D}}^{[\perp\mathcal{K}]}) \geq \kappa$ is necessary for the existence of $\mathcal{S}_{\kappa}^{p \times m}(\Pi_+)$ -functions S analytic on $\sigma(Z) \cup \sigma(W)$ satisfying the interpolation conditions (1.11), (1.12), (1.13).

Proof of necessity for the generalized Schur-class setting. If $S \in \mathcal{S}_{\kappa'}^{p \times m}(\Pi_+)$ is a solution of the interpolation conditions with $\kappa' \leq \kappa$, then as in Section 5.1, there is a $m \times m$ -matrix function Ψ with $\det \Psi$ having no zeros in $\sigma(Z) \cup \sigma(W)$ and having κ zeros in Π_+ so that the subspace $\mathcal{G}_S := \begin{bmatrix} S \\ I_m \end{bmatrix} \psi^{-1} \Psi H_m^2(\Pi_+)$ satisfies the inclusion (5.16). We note that then \mathcal{G}_S is a negative subspace of \mathcal{K} and the fact that Ψ has κ zeros means that \mathcal{G}_S has codimension κ in a maximal negative subspace of $\mathcal{K} := \begin{bmatrix} L_p^2(i\mathbb{R}) \\ H_m^2(\Pi_+) \end{bmatrix}$. As \mathcal{G}_S is also a subspace of $\mathcal{M}_{\mathfrak{D}}$, it follows by Lemma 5.2 that the negative signature of $\mathcal{M}_{\mathfrak{D}}^{[\perp\mathcal{K}]}$ must be at least κ . Thus $\nu_-(\mathcal{M}_{\mathfrak{D}}^{[\perp\mathcal{K}]}) \geq \kappa$ is necessary for the existence of a solution S of the interpolation problem in the class $\mathcal{S}_{\kappa}^{p \times m}(\Pi_+)$. As part of the sufficiency direction, we shall show that conversely, if $\kappa = \nu_-(\mathcal{M}_{\mathfrak{D}}^{[\perp\mathcal{K}]})$, then we can always find solutions S of the interpolation conditions in the class $\mathcal{S}_{\kappa}^{p \times m}(\Pi_+)$. This establishes the formula

in statement (2) of Theorem 5.1 as the minimal κ such that solutions of the interpolation conditions can be found in class $\mathcal{S}_\kappa^{p \times m}(\Pi_+)$.

Proof of sufficiency for the generalized Schur-class setting. Let us suppose that $\Gamma_{\mathfrak{D}}$ is invertible and hence that $\mathcal{M}_{\mathfrak{D}}$ is a regular subspace of the Kreĭn space $L_{p+m}^2(i\mathbb{R})$ with the $J = \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix}$ -inner product. By the results of [10], there is a J -phase function Θ so that the Beurling-Lax representation (2.29) holds (we avoid using the formula (2.17) for Θ at this stage). We now assume that $\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]}$ has negative signature $\nu_-(\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]})$ equal to κ . We wish to verify the linear-fractional parametrization (5.1)–(5.2) for the set of all $\mathcal{S}_\kappa^{p \times m}(\Pi_+)$ -class solutions of the interpolation conditions.

Suppose first that S is any $\mathcal{S}_\kappa^{p \times m}(\Pi_+)$ -class solution of the interpolation conditions. By the graph-space criterion for such solutions, there is a $m \times m$ -matrix valued function Ψ analytic on Π_+ with $\det \Psi$ having κ zeros but none in $\sigma(Z) \cup \sigma(W)$, so that (5.16) holds. But then

$$\mathcal{G}_S := \begin{bmatrix} S \\ I_m \end{bmatrix} \psi^{-1} \Psi H_m^2(\Pi_+)$$

is a shift-invariant negative subspace of \mathcal{K} contained in $\mathcal{M}_{\mathfrak{D}}$ and having codimension κ in a maximal negative subspace of \mathcal{K} . It now follows from Lemma 5.2 that \mathcal{G}_S is maximal negative as a subspace of $\mathcal{M}_{\mathfrak{D}}$. As \mathcal{G}_S is also shift-invariant and multiplication by Θ is a Kreĭn-space isomorphism from $H_{p+m}^2(\Pi_+)$ onto $\mathcal{M}_{\mathfrak{D}}$, it follows that \mathcal{G}_S is the image under multiplication by Θ of a shift-invariant J -maximal negative subspace of $H_{p+m}^2(\Pi_+)$, i.e.,

$$\mathcal{G}_S := \begin{bmatrix} S \\ I_m \end{bmatrix} \cdot \psi^{-1} \Psi \cdot H_m^2(\Pi_+) = \Theta \cdot \begin{bmatrix} G \\ I_m \end{bmatrix} \cdot H_m^2(\Pi_+) \quad (5.20)$$

for a $\mathcal{S}^{p \times m}(\Pi_+)$ -class function G . From the fact that Ψ has no zeros in $\sigma(Z) \cup \sigma(W)$ one can read off from (5.20) that $\psi(\Theta_{21}G + \Theta_{22})$ has no zeros in $\sigma(Z) \cup \sigma(W)$ and from the representation (5.20) the linear-fractional representation (5.1) follows as well. From the subspace identity (5.20) one can also read off that there is a $m \times m$ matrix function Q with $Q^{\pm 1} \in H_{m \times m}^\infty(\Pi_+)$ such that

$$S\psi^{-1}\Psi = (\Theta_{11}G + \Theta_{12})Q \quad \text{and} \quad \psi^{-1}\Psi = (\Theta_{21}G + \Theta_{22})Q.$$

Solving the second equation for Q then gives

$$Q = (\Theta_{22}G + \Theta_{22})^{-1}\psi^{-1}\Psi.$$

Substituting this back into the first equation and then solving for S leads to the linear-fractional representation (5.1) for S .

Let now G be any Schur-class function satisfying the additional constraint (5.2). Since multiplication by Θ is a Kreĭn-space isomorphism from $H_{p+m}^2(\Pi_+)$ to $\mathcal{M}_{\mathfrak{D}}$ and $\begin{bmatrix} G \\ I_m \end{bmatrix} H_m^2(\Pi_+)$ is a maximal negative shift-invariant subspace of $\mathcal{M}_{\mathfrak{D}}$, it follows that $\Theta \cdot \begin{bmatrix} G \\ I_m \end{bmatrix} H_m^2(\Pi_+)$ is maximal negative as a subspace of $\mathcal{M}_{\mathfrak{D}}$. By Lemma 5.2, it follows that $\Theta \cdot \begin{bmatrix} G \\ I_m \end{bmatrix} H_m^2(\Pi_+)$ has codimension $\kappa = \nu_-(\mathcal{M}_{\mathfrak{D}}^{[\perp \mathcal{K}]})$ in a maximal negative subspace of \mathcal{K} . As

$\Theta \cdot \begin{bmatrix} G \\ I_m \end{bmatrix} H_m^2(\Pi_+)$ is also shift-invariant, it follows that there must be a contractive matrix function S on the unit circle and a bounded analytic $m \times m$ -matrix function Ψ on Π_+ such that Ψ has exactly κ zeros in Π_+ and Ψ is bounded and invertible on $i\mathbb{R}$ so that

$$\begin{bmatrix} S \\ I_m \end{bmatrix} \cdot \psi^{-1}\Psi \cdot H_m^2(\Pi_+) = \Theta \cdot \begin{bmatrix} G \\ I \end{bmatrix} \cdot H_m^2(\Pi_+). \quad (5.21)$$

In particular, $\psi^{-1}\Psi \cdot H_m^2(\Pi_+) \subset (\Theta_{21}G + \Theta_{22}) \cdot H_m^2(\Pi_+)$, so there is a $Q \in H_{p+m}^\infty(\Pi_+)$ so that $\psi^{-1}\Psi = (\theta_{21}G + \Theta_{22})Q$, i.e., so that

$$\Psi = \psi(\Theta_{21}G + \Theta_{22})Q.$$

As $\psi(\Theta_{21}G + \Theta_{22})$ has no zeros in $\sigma(Z) \cup \sigma(W)$ by assumption, it follows that none of the zeros of Ψ are in $\sigma(Z) \cup \sigma(W)$. By the criterion (5.16) for $\mathcal{S}_{\kappa'}^{p \times m}(\Pi_+)$ -class solutions of the interpolation conditions with $\kappa' \leq \kappa$, we read off from (5.21) that S so constructed is a $\mathcal{S}_{\kappa'}^{p \times m}(\Pi_+)$ -class solution of the interpolation conditions for some $\kappa' \leq \kappa$. However, from the proof of the necessity direction already discussed, it follows that necessarily $\kappa' \geq \kappa$. Thus S so constructed is a $\mathcal{S}_{\kappa}^{p \times m}(\Pi_+)$ -class solution of the interpolation conditions. The subspace identity (5.21) leads to the formula (5.1) for S in terms of G just as in the previous paragraph.

Remark 5.3. We conclude that the Grassmannian approach extends to the generalized Schur-class setting. As in the classical Schur-class case, one can avoid the elaborate winding-number argument used in Section 5.1 by using Kreĭn-space geometry (namely, the fact the a Kreĭn-space isomorphism maps maximal negative subspaces to maximal negative subspaces combined with Lemma 5.2), unlike the story for the Fundamental Matrix Inequality Potapov approach, which avoids the winding number argument in an elegant way for the definite case but appears to still require such an argument for the indefinite generalized Schur-class setting.

5.5. State-space versus Grassmannian/Kreĭn-space-geometry solution criteria in the generalized Schur-class setting. The work of the previous subsections shows that each of conditions (1) and (2) in Theorem 5.1 is equivalent to the existence of $\mathcal{S}_{\kappa}^{p \times m}(\Pi_+)$ -class solutions of the interpolation conditions (1.11)–(1.13), and that condition (2) is equivalent to condition (1). It follows that conditions (1), (2), (3) are all equivalent to each other. Here we wish to see this latter fact directly in a more concrete form, analogously to what is done in Section 4 above for the classical Schur-class setting.

As in Section 4, we impose an assumption a little stronger than the condition that $\Gamma_{\mathfrak{D}}$ be invertible, namely, the Nondegeneracy Assumption: $\mathcal{M}_{\mathfrak{D}}$, $\mathcal{M}_{\mathfrak{D}} \cap H_{p+m}^2(\Pi_+)$, and $\mathcal{M}_{\mathfrak{D}} \cap H_{p+m}^2(\Pi_-)$ are all regular subspaces of $L_{p+m}^2(i\mathbb{R})$ (with the $J (= \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix})$ -inner product). Then Lemmas 4.2 and 4.3 go through with no change. Lemma 4.4 goes through, but with the

in particular statement generalized to the following (here $\nu_-(\mathcal{L})$ refers to negative signature of the given subspace \mathcal{L} of $L_{p+m}^2(i\mathbb{R})$ with respect to the J -inner product):

- In particular, $\nu_-(\mathcal{M}_{\mathfrak{D}}^{[\perp\mathcal{K}]}) = \kappa$ if and only if

$$\nu_-(\mathcal{M}_{\mathfrak{D}}^{[\perp\mathcal{K}]}_0) = \kappa$$

if and only if

$$\nu_-(\mathbb{G}_{T^{[*]}}) + \nu_-(\mathcal{M}_{\mathfrak{D}}^{[\perp]}_1) = \kappa.$$

Lemma 4.5 has the more general form:

- (1) $\nu_-(\mathbb{G}_{T^{[*]}}) = \kappa$ if and only if $\nu_-(I + TT^{[*]}) = \kappa$ (where $I + TT^{[*]}$ is considered as an operator on $\text{Ran } J(\mathcal{C}_{Z, [X - Y]})^*$).
- (2) $\nu_-(\mathcal{M}_{\mathfrak{D}}^{[\perp]}_1) = \nu_-(\text{Ran } \mathcal{O}_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right], W})$.
- (3) $\nu_-(\widehat{\mathbb{G}}_T) = \nu_-(I + T^{[*]}T)$ (where $I + T^{[*]}T$ is considered as an operator on $\text{Ran } \mathcal{O}_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right], W}$).
- (4) $\nu_+(\mathcal{M}_{\mathfrak{D}, 1}) = \nu_-(\text{Ran } J(\mathcal{C}_{Z, [X - Y]})^*)$.

Lemma 4.6 is already in general form but its corollary, namely Lemma 4.7, can be given in a more general form:

- The following conditions are equivalent:
 - (1) $\nu_-(\mathbf{\Gamma}_{\mathfrak{D}}) = \kappa$.
 - (2) $\nu_+(\text{Ran } \mathcal{O}_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right], W}) + \nu_-(\mathbb{G}_{T^{[*]}}) = \kappa$.
 - (3) $\nu_-(\text{Ran } J(\mathcal{C}_{Z, [X - Y]})^*) + \nu_+(\widehat{\mathbb{G}}_T) = \kappa$.

Putting the pieces together, we have the following chain of reasoning. By the generalized version of Lemma 4.4, we have

$$\nu_-(\mathcal{M}_{\mathfrak{D}})^{[\perp\mathcal{K}]} = \nu_-(\mathbb{G}_{T^{[*]}}) + \nu_-(\mathcal{M}_{\mathfrak{D}}^{[\perp]}_1) \quad (5.22)$$

where, by the generalized version of Lemma 4.5 part (2),

$$\nu_-(\mathcal{M}_{\mathfrak{D}}^{[\perp]}_1) = \nu_-(\text{Ran } \mathcal{O}_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right], W}).$$

Thus (5.22) becomes

$$\nu_-(\mathcal{M}_{\mathfrak{D}})^{[\perp\mathcal{K}]} = \nu_-(\mathbb{G}_{T^{[*]}}) + \nu_-(\text{Ran } \mathcal{O}_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right], W}).$$

By (1) \Leftrightarrow (2) in the generalized Lemma 4.6, we get

$$\nu_-(\mathcal{M}_{\mathfrak{D}})^{[\perp\mathcal{K}]} = \nu_-(\mathbf{\Gamma}_{\mathfrak{D}})$$

which gives us (1) \Leftrightarrow (2) in Theorem 5.1.

To give a direct proof of (1) \Leftrightarrow (3) in Theorem 5.1, we note the concrete identification (4.25) of the space $\mathcal{H}(K_{\Theta, J})$ (with J -inner product on

$\text{Ran}(P^J H_{p+m}^2(\Pi_+) - P_{\mathcal{M}_{\mathfrak{D}}}^J)$ which again leads to the more compact identification (4.26) from which we immediately see that

$$\nu_-(\mathcal{H}(K_{\Theta,J})) = \nu_-(\text{Ran } J(\mathcal{C}_{Z,[X-Y]})^*) + \nu_+(\widehat{\mathbb{G}}_T).$$

By (1) \Leftrightarrow (3) in the generalized Lemma 4.7, this last expression is equal to $\nu_-(\mathbf{\Gamma}_{\mathfrak{D}})$, and we have our more concrete direct proof of the equivalence of conditions (1) and (3) in Theorem 5.1.

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